Combinatorial description of facets of the cone of exact games (or coherent lower probabilities)

Milan Studený

presenting joint work with Václav Kratochvíl

Czech Academy of Sciences
Institute of Information Theory and Automation
Prague, Czech Republic

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Summary of the talk

Abstract: exact games and coherent lower probabilities

The 1972 concept of an *exact (transferable utility coalitional) game* by Schmeidler corresponds to the concept of a *coherent lower probability*, studied in the context of imprecise probabilities.

- D. Schmeidler. Cores of exact games, I. *Journal of Mathematical Analysis and Applications*, 40:214–225, 1972.
- P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, 1991.

In our recent paper we characterized the *facets of the cone of exact games*, that is, the minimal set of linear inequalities defining this cone.

- M. Studený, V. Kratochvíl. Facets of the cone of exact games. *Mathematical Methods of Operations Research*, 95(1):35–80, 2022.

The result offers a direct linear computational test to decide whether a given lower probability (on a small sample space) is coherent or not.

Abstract: combinatorial interpretation of the facets

The facet-defining inequalities for the exact cone appear to have *combinatorial interpretation*: they correspond to certain set systems (= systems of coalitions, that is, of subsets of the sample space).

This talk is about the main result of our 2022 paper, in which the characterization of these set systems as the so-called *indecomposable min-semi-balanced systems* was brought.

This concept of a semi-balanced set system (we use) generalizes the classic game-theoretical concept of a balanced set system.

L.S. Shapley. On balanced sets and cores. *Naval Research Logistics Quarterly*, 14:453–460, 1967.

We also introduced *pictorial representatives of semi-balanced systems* and created a web catalogue of all indecomposable min-semi-balanced systems for at most 6 players (= for a sample space of cardinality at most 6).

Basic notation and concept of a coalitional game

Let N be a non-empty finite *basic set*.

In the game-theoretical context, the elements of the basic set N correspond to *players* and (non-empty) subsets of N to *coalitions*.

The symbol $\mathcal{P}(N) \; := \; \{S : S \subseteq N\}$ will denote its power set.

The symbol $\mathbb{R}^N$ will be used to denote the Euclidean space of real vectors $[x_i]_{i \in N}$ whose components are indexed by elements of N.

Definition (a coalitional game)

A (*transferable-utility coalitional*) *game over* N is modeled by a real function $m \colon \mathcal{P}(N) \to \mathbb{R}$ such that $m(\emptyset) = 0$, which is sometimes named the *characteristic function of the game*. The game m is called

- *non-negative* if $m(A) \geq 0$ for any $A \subseteq N$,
- *standardized* if $m(\{i\}) = 0$ for any $i \in N$,
- *normalized* if $m(N) = 1$.

Core of a coalitional game and its profit interpretation

Definition (core of a game)

The *core* $C(m)$ of a game m over N is a polyhedron in $\mathbb{R}^N$ defined by

$$C(m) := \{\, [x_i]_{i\in N} \in \mathbb{R}^N \,:\, \textstyle\sum_{i\in N} x_i = m(N) \;\&\; \textstyle\sum_{i\in A} x_i \geq m(A) \text{ for any } A \subseteq N \}\,.$$

Its *profit interpretation* implicitly assumes that m is a non-negative game.

- The value $m(N)$ for the so-called *grand coalition N* is interpreted as the amount of money the players altogether won and wish to divide.
- The values $m(A)$ for a coalition $A \subseteq N$ is interpreted as the *strength* of the coalition A. If the total win $m(N)$ is divided among individual players, then members of any coalition A should together obtain at least what is the strength $m(A)$.
- Vectors in $C(m)$ are interpreted as *payoff vectors* compatible with the basic requirements, dictated by the strengths of the coalitions.

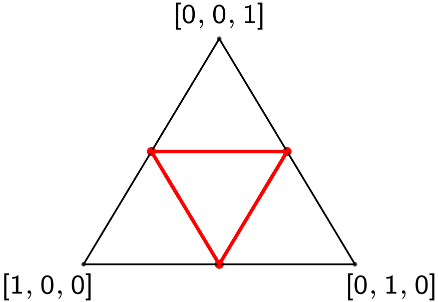
Simple illustration of the concept of a core polytope

Take $N := \{a, b, c\}$ and

A	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
$m(A)$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

In this particular case, because of $x_a + x_b + x_c = m(N) = 1$ one has $x_b + x_c \geq m(\{b, c\}) = \frac{1}{2} \;\Leftrightarrow\; x_a \leq 1 - m(\{b, c\}) = 1 - \frac{1}{2} = \frac{1}{2}$. For this reason

$$C(m) \;=\; \{\, [x_a, x_b, x_c] \,:\; x_a + x_b + x_c = 1 \;\&\; 0 \leq x_i \leq \frac{1}{2} \quad \text{for } i \in N \,\} \,.$$



Game-theoretical notions: balanced and exact game

Definition (balanced and exact game)

We say that a game m over N is *balanced* if $C(m) \neq \emptyset$.

A game m over N is *exact* if, for each coalition $S \subseteq N$, a vector $[x_i]_{i\in N} \in C(m)$ exists such that $\sum_{i\in S} x_i = m(S)$.

It is immediate that every exact game is balanced.
One can also observe that any *non-negative exact game* is *monotone*:

$$A \subseteq B \subseteq N \quad \Rightarrow \quad m(A) \leq m(B)\,.$$

This is because m is superadditive: $m(S) + m(T) \leq m(S \cup T)$ if $S \cap T = \emptyset$.
To this end take $x \in C(m)$ with $\sum_{i\in S\cup T} x_i = m(S \cup T)$ and have both $\sum_{i\in S} x_i \geq m(S)$ and $\sum_{i\in T} x_i \geq m(T)$.

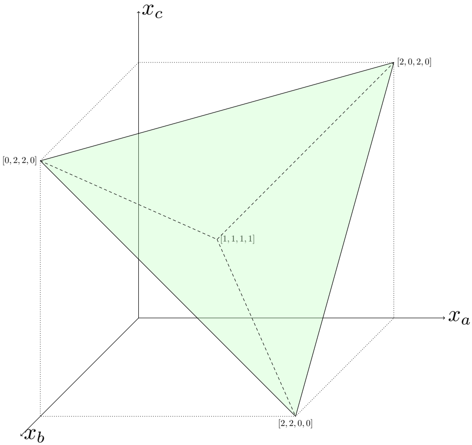
An example of (the core of) an exact game

$N = \{a, b, c, d\}$ $\quad m(N) = 4, \quad m(\{a, b, c\}) = 3,$

$$m(\{a, b, d\}) = m(\{a, c, d\}) = m(\{b, c, d\}) = 2,$$

$$m(\{a, b\}) = m(\{a, c\}) = m(\{b, c\}) = 2, \quad m(S) = 0 \text{ for other } S \subseteq N.$$

$$x_d = 4 - x_a - x_b - x_c$$



	x_a	x_b	x_c	x_d
α	1	1	1	1
β	2	2	0	0
γ	2	0	2	0
δ	0	2	2	0

Lower probability

In the context of imprecise probabilities, the basic set N is interpreted as the *sample space* for the considered random variables. It is also called the frame of discernment in context of Dempster-Shafer theory of evidence.

The following was taken over from the October 2022 SIPTA seminar presentation `Coalitional game theory VS Imprecise probability: two sides of the same coin` by Ignacio Montes.

Definition (lower probability)

A *lower probability* on N is a function $\underline{P}\colon \mathcal{P}(N) \to [0,1]$ such that

- $\underline{P}(\emptyset) = 0$ and $\underline{P}(N) = 1$,
- $\underline{P}(A) \leq \underline{P}(B)$ whenever $A \subseteq B \subseteq N$.

Taking $\underline{P}(A) := m(A)$ for any $A \subseteq N$ yields (with a previous observation):

$$\text{a normalized monotone game over } N \quad \Leftrightarrow \quad \text{lower probability on } N,$$

$$\text{a normalized non-negative exact game} \quad \Rightarrow \quad \text{lower probability on } N.$$

Credal set and coherent lower probability

Definition (credal set, coherent lower probability)

Given a lower probability $\underline{P}$ on N, the respective *credal set* is

$$\mathcal{M}(\underline{P}) := \{\, P \text{ a probability measure on } N \,:\, P(A) \geq \underline{P}(A) \ \text{ for } A \subseteq N \,\}.$$

One says that a lower probability $\underline{P}$ on N

- *avoids sure loss* if $\mathcal{M}(\underline{P}) \neq \emptyset$,
- is *coherent* if $\underline{P}(A) = \min \{\, P(A) \,:\, P \in \mathcal{M}(\underline{P}) \,\}$ for any $A \subseteq N$.

Given a normalized monotone game m over N, interpret any $x \in C(m)$ as a probability distribution P given by $P(A) := \sum_{i \in A} x_i$ for $A \subseteq N$ and get :

$$\begin{aligned} \text{core polytope } C(m) \quad &\Leftrightarrow \quad \text{credal set } \mathcal{M}(\underline{P}) \text{ with } \underline{P} := m, \\ m \text{ is a balanced game} \quad &\Leftrightarrow \quad \underline{P} := m \text{ avoids sure loss}, \\ m \text{ is an exact game} \quad &\Leftrightarrow \quad \underline{P} := m \text{ is a coherent lower probability.} \end{aligned}$$

So, a normalized non-negative exact game $\Leftrightarrow$ a coherent lower probability.

Cones of balanced and exact games

Definition (polyhedral cone)

Recall that a *polyhedral cone* is a subset of an Euclidean space $\mathbb{R}^{\mathcal{I}}$ specified by finitely many inequalities $\langle x, y \rangle \geq 0$ for $x \in \mathbb{R}^{\mathcal{I}}$, where $y \in R^{\mathcal{I}}$ and $\langle x, y \rangle := \sum_{i \in \mathcal{I}} x_i \cdot y_i$ is the scalar product.

Further considerations concern more-dimensional Euclidean space $\mathbb{R}^{\mathcal{P}(N)}$, whose elements are vectors with components indexed by subsets of N, interpreted as *set functions over N*.

Important classes of models in cooperative game theory are:

- the class of *balanced games* $\mathcal{B}(N)$,
- the class of *exact games* $\mathcal{E}(N)$.

They are both polyhedral cones in $\mathbb{R}^{\mathcal{P}(N)}$, related as follows:

$$\mathcal{E}(N) \subseteq \mathcal{B}(N) \subseteq \mathbb{R}^{\mathcal{P}(N)} \qquad \text{with strict inclusion if } |N| \geq 3\,.$$

Further two classes of games will be mentioned in the end of the talk.

Why non-negativity assumption is not restrictive

Definition (linearity space)

The *linearity space* for a (polyhedral) cone $C \subseteq \mathbb{R}^{\mathcal{I}}$ is the intersection $(-C) \cap C$, where $(-C) := \{\, x \in \mathbb{R}^{\mathcal{I}} \,:\, (-x) \in C \,\}$.

The linearity space for both $\mathcal{B}(N)$ and $\mathcal{E}(N)$ is the space of *additive games*

$$\{\, m \in \mathbb{R}^{\mathcal{P}(N)} \,:\, m(A \cup B) = m(A) + m(B) \quad \text{if } A, B \subseteq N,\ A \cap B = \emptyset \,\}\,.$$

Every polyhedral set has unique inclusion minimal linear inequality description by the so-called *facet-defining inequalities*. (= in its ambient space, where the uniqueness of an inequality is up to its positive multiple)

Additive games have the form $m(A) := \sum_{i \in A} y_i$ for $A \subseteq N$, where $y \in \mathbb{R}^N$. This implies that any exact game can be turned into a standardized exact game by adding an additive game (*standardization*). That game is then non-negative.

Why the minimal inequality descriptions are equivalent

The above observation implies that the facet-defining inequalities for the cone $\mathcal{E}(N)$ and the cone of *standardized exact games* are the same, one just adds the constraints $m(\{i\}) = 0$ for $i \in N$ is the latter case.

The same consideration works with the cone of *non-negative exact games,* where one only adds inequality constraints $m(\{i\}) \geq 0$ for $i \in N$.

The case of *coherent lower probabilities*, that is, of *normalized non-negative exact games*, also fits in this frame. In this case one just adds the normalization constraint $m(N) = 1$ and the non-negativity constraints $m(\{i\}) \geq 0$ for $i \in N$.

Completely analogous considerations work for the cone of *balanced games* and the class of lower probabilities avoiding sure loss.

This also works for two other classes of coalitional games to be mentioned later, namely the *supermodular/convex* games and the *totally balanced* games.

Example: three basic variables

Assume the case $N := \{\, a, b, c \,\}$.

In this case, the cone $\mathcal{B}(N)$ of *balanced games* $m \in \mathbb{R}^{\mathcal{P}(N)}$ is specified by five (facet-defining) inequalities, which fall into 3 permutational types:

- $m(\{a\}) + m(\{b\}) + m(\{c\}) \le m(\{a,b,c\}) + 2 \cdot m(\emptyset)$ [1×]
- $m(\{a\}) + m(\{b,c\}) \le m(\{a,b,c\}) + m(\emptyset)$ [3×]
- $m(\{a,b\}) + m(\{a,c\}) + m(\{b,c\}) \le 2 \cdot m(\{a,b,c\}) + m(\emptyset)$ [1×]

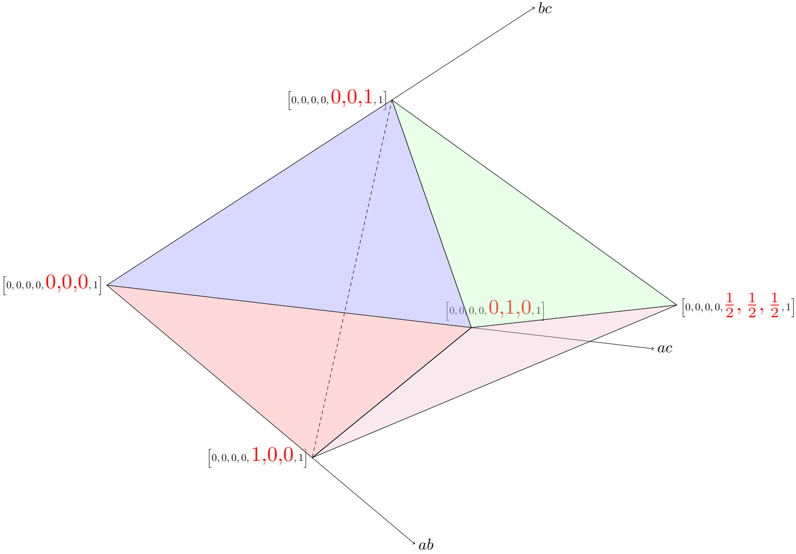
On the other hand, the cone $\mathcal{E}(N)$ of *exact games* $m \in \mathbb{R}^{\mathcal{P}(N)}$ is specified by six (facet-defining) inequalities of 2 permutational types:

- $m(\{a\}) + m(\{b\}) \le m(\{a,b\}) + m(\emptyset)$ [3×]
- $m(\{a,c\}) + m(\{b,c\}) \le m(\{a,b,c\}) + m(\{c\})$ [3×]

The inequalities are intentionally written in the way one can possibly recognize their hidden combinatorial structure: they correspond to particular set systems.

$N = \{a, b, c\}$: normalized standardized exact games

Legend: $\left[m(\emptyset), m(\{a\}), m(\{b\}), m(\{c\}), m(\{a,b\}), m(\{a,c\}), m(\{b,c\}), m(\{a,b,c\})\right]$



Facet-defining inequalities for the cone of balanced games

were characterized by Shapley on basis of former results by Bondareva.

L.S. Shapley. On balanced sets and cores. *Naval Research Logistics Quarterly*, 14:453–460, 1967.

O. Bondareva. Some applications of linear programming methods to the theory of cooperative games. *Problemy Kibernetiki*, 10:119-139, 1963.

They correspond to "minimal balanced collections" of subsets of N whose union is N. In our example (= the case $N := \{ a, b, c \}$), the inequality

- $m(\{a\}) + m(\{b\}) + m(\{c\}) \leq m(\{a, b, c\}) + 2 \cdot m(\emptyset)$

corresponds to the set system $\mathcal{S} := \{ \{a\}, \{b\}, \{c\} \}$, while

- $m(\{a\}) + m(\{b, c\}) \leq m(\{a, b, c\}) + m(\emptyset)$

corresponds to the set system $\mathcal{S} := \{ \{a\}, \{b, c\} \}$, and

- $m(\{a, b\}) + m(\{a, c\}) + m(\{b, c\}) \leq 2 \cdot m(\{a, b, c\}) + m(\emptyset)$

corresponds to $\mathcal{S} := \{ \{a, b\}, \{a, c\}, \{b, c\} \}$.

Some specific notation and terminology

Let N be our non-empty finite *basic set*.

Recall that $\mathcal{P}(N) := \{S : S \subseteq N\}$ denotes *its power set*.

Given $\mathcal{S} \subseteq \mathcal{P}(N)$, $\bigcup \mathcal{S}$ will denote the union, $\bigcap \mathcal{S}$ the intersection.

Given $S \subseteq N$, the symbol χ_S will denote the *incidence vector* of S in $\mathbb{R}^N$, that is, its zero-one indicator defined by

$$(\chi_S)_i := \begin{cases} 1 & \text{for } i \in S, \\ 0 & \text{for } i \in N \setminus S, \end{cases} \qquad \text{whenever } i \in N.$$

Any vector whose components equal each other, that is, a vector of the form $[r, \ldots, r] \in \mathbb{R}^N$ where $r \in \mathbb{R}$, will be called a *constant vector* in $\mathbb{R}^N$. A special case is the *zero vector*, whose components are zeros.

Classic (minimal) balanced set systems

Any subset $\mathcal{S}$ of $\mathcal{P}(N)$ is called a *set system*.

Definition (non-trivial, balanced, and min-balanced set system)

Let $M \subseteq N$ be a subset of the basic set N with $|M| \geq 2$.
A non-empty system $\mathcal{S}$ of subsets of M such that $\emptyset, M \notin \mathcal{S}$ will be called a *non-trivial* set system *on M*.

A *balanced* system (on M) is a non-trivial set system $\mathcal{B}$ on M such that χ_M is a conic combination of $\{\chi_S : S \in \mathcal{B}\}$ with all coefficients non-zero.

A balanced system $\mathcal{B}$ on M is called *minimal* if there is no balanced system $\mathcal{C}$ on M with $\mathcal{C} \subset \mathcal{B}$; $\mathcal{B}$ will then be called briefly *min-balanced* (on M).

An important fact is that the vectors $\{\, \chi_S : S \in \mathcal{B}\,\}$ for a min-balanced set system $\mathcal{B}$ are linearly independent. (this is a characterization of minimality)

Example If $N = \{a, b, c\}$, the set system $\mathcal{S} := \{\, \{a, b\}, \{a, c\}, \{b, c\}\,\}$ is min-balanced system on N: one has $\frac{1}{2} \cdot \chi_{\{a,b\}} + \frac{1}{2} \cdot \chi_{\{a,c\}} + \frac{1}{2} \cdot \chi_{\{b,c\}} = \chi_N$.

An inequality assigned to a min-balanced set system

Definition (inequality assigned to a min-balanced set system)

Any min-balanced system $\mathcal{B}$ on $M = \bigcup \mathcal{B} \subseteq N$ defines the next inequality:

$$\sum_{S \in \mathcal{B}} \lambda_S \cdot m(S) \leq m(\bigcup \mathcal{B}) \; + \; (-1 + \sum_{S \in \mathcal{B}} \lambda_S) \cdot m(\emptyset) \qquad (1)$$

for $m \in \mathbb{R}^{\mathcal{P}(N)}$, where $\lambda_S > 0$, $S \in \mathcal{B}$, are uniquely determined (positive) balancing coefficients in the decomposition $\sum_{S \in \mathcal{B}} \lambda_S \cdot \chi_S = \chi_{\bigcup \mathcal{B}}$ of χ_M.

Example $\mathcal{S} := \{\, \{a,b\}, \{a,c\}, \{b,c\} \,\}$ gives
$\frac{1}{2} \cdot m(\{a,b\}) + \frac{1}{2} \cdot m(\{a,c\}) + \frac{1}{2} \cdot m(\{b,c\}) \leq m(N) \; + \; \frac{1}{2} \cdot m(\emptyset)$

Shapley-Bondareva theorem (the balanced cone characterization)

The facet-defining inequalities for the cone $\mathcal{B}(N)$ are just the inequalities (1) for min-balanced set systems $\mathcal{B}$ on (whole) N.

Remarks on the characterization of balanced games

One of the classic results in cooperative game theory from the 1960s is an iterative algorithm, which for every N, $|N| \geq 2$, generates the complete list of all min-balanced set systems $\mathcal{B}$ on N.

B. Peleg. An inductive method for constructing minimal balanced collections of finite sets. *Naval Research Logistics Quarterly*, 12:155–162, 1965.

As explained earlier, Shapley-Bondareva theorem gives a linear inequality characterization of *lower probabilities avoiding sure loss*.

If one combines this with the procedure offered by Peleg it may lead to a simple computational test (for a small sample space N) of whether a given lower probability avoids sure loss or not.

The case of exact games (viewed as an analogy)

The main topic of the talk is the characterization of facet-defining inequalities for the cone $\mathcal{E}(N)$ of exact games.

In a sense, our result is analogous to the classical result of Shapley and Bondareva for $\mathcal{B}(N)$, but most of the concepts we need are slightly more complicated (though not by much).

As explained earlier, this result yields linear inequality characterization of *coherent lower probabilities* (for a small sample space N).

The basic step is the following generalization.

Definition (semi-conic combination)

A linear combination $\sum_{i\in I} \lambda_i \cdot x_i$ in $\mathbb{R}^N$ will be called *semi-conic* if at most one of its coefficients λ_i is strictly negative: $|\{\, j \in I \,:\, \lambda_j < 0\}| \leq 1$.

Any conic combination is semi-conic, but the mathematical properties of the operator of semi-conic closure are different (not idempotent).

The concept of a (minimal) semi-balanced set system

Definition (semi-balanced and min-semi-balanced set system)

Assume that N is a finite set with $|N| \geq 2$. We shall say that a non-trivial set system $\mathcal{S}$ on N is *semi-balanced* (on N) if there is a constant vector in $\mathbb{R}^N$ which is a semi-conic combination of incidence vectors $\{\chi_S \,:\, S \in \mathcal{S}\}$ with all coefficients non-zero.

A semi-balanced system $\mathcal{S}$ on N will be called *minimal* if there is no semi-balanced system $\mathcal{C}$ on N with $\mathcal{C} \subset \mathcal{S}$. We will then say briefly that such a set system is *min-semi-balanced* (on N).

Important fact is that, in case of a min-semi-balanced set system, the vectors $\{\, \chi_S \,:\, S \in \mathcal{S}\,\}$ are affinely independent.

So, an affine semi-conic combination yielding a constant vector is unique.
(affine combination = a linear combination where the sum of the coefficients is 1)

Example: semi-balanced systems which are not balanced

Clearly, every balanced system on N is semi-balanced on N.

The union of two semi-balanced systems need not be semi-balanced.
(unlike the case of balanced systems, which are closed under union)

Example (two semi-balanced set systems)

Consider $N := \{a, b, c, d\}$ and set systems

$\mathcal{S} := \{\, \{a\}, \{b\}, \{a, b\} \,\}$ and $\mathcal{T} := \{\, \{c\}, \{b, c\}, \{a, c, d\} \,\}$.

The equalities

$$1 \cdot \chi_a + 1 \cdot \chi_b + (-1) \cdot \chi_{ab} = 0 \qquad (-1) \cdot \chi_c + 1 \cdot \chi_{bc} + 1 \cdot \chi_{acd} = \chi_N$$

imply that $\mathcal{S}$ and $\mathcal{T}$ are (both) min-semi-balanced set systems on N.
Their union $\mathcal{D} := \mathcal{S} \cup \mathcal{T}$ is not semi-balanced on N. (details are omitted)

An inequality assigned to a min-semi-balanced system

Definition (inequality assigned to a min-semi-balanced set system)

Every min-semi-balanced system $\mathcal{S}$ on N defines the next inequality:

$$\sum_{S\in\mathcal{S}} \lambda_S \cdot m(S) \;\leq\; r \cdot m(N) \;+\; (-r + \sum_{S\in\mathcal{B}} \lambda_S) \cdot m(\emptyset) \qquad (2)$$

for $m \in \mathbb{R}^{\mathcal{P}(N)}$, where $\sum_{S\in\mathcal{S}} \lambda_S \cdot \chi_S$ is the unique affine semi-conic combination yielding a constant vector $[r, \ldots, r] \in \mathbb{R}^N$.

If $\mathcal{S} = \mathcal{B}$ is min-balanced then (2) is a multiple of the former inequality (1).

Example for $\mathcal{S} := \{\,\{a,b\},\{a,c\},\{b,c\}\,\}$ (2) gives
$\frac{1}{3} \cdot m(\{a,b\}) + \frac{1}{3} \cdot m(\{a,c\}) + \frac{1}{3} \cdot m(\{b,c\}) \leq \frac{2}{3} \cdot m(N) \;+\; \frac{1}{3} \cdot m(\emptyset)$
while the formerly assigned inequality (to a min-balanced system) was
$\frac{1}{2} \cdot m(\{a,b\}) + \frac{1}{2} \cdot m(\{a,c\}) + \frac{1}{2} \cdot m(\{b,c\}) \leq 1 \cdot m(N) \;+\; \frac{1}{2} \cdot m(\emptyset)$.

Characterization of exact games

This is an analogy of Shapley-Bondareva theorem, but a complete one.

Proposition (the exact cone characterization)

The inequalities (2) for min-semi-balanced systems on N characterize the cone $\mathcal{E}(N)$, but some of them might be superfluous (if $|N| \geq 3$).

To give examples of facet-defining inequalities for $\mathcal{E}(N)$ (= those which are not superfluous), consider again the case $N := \{\, a, b, c \,\}$. The inequality

- $m(\{a\}) + m(\{b\}) \leq m(\{a, b\}) + m(\emptyset)$

is a re-arranged version of the one assigned the min-semi-balanced system $\mathcal{S} := \{\, \{a\}, \{b\}, \{a, b\} \,\}$ on N (with $r = 0$). Note that $\boxed{\mathcal{S} = \mathcal{B} \cup \{M\}}$, where $\mathcal{B} = \{\, \{a\}, \{b\} \,\}$ is a min-balanced system on $M := \{a, b\}$.

Another facet-defining inequality mentioned earlier was

- $m(\{a, c\}) + m(\{b, c\}) \leq m(\{a, b, c\}) + m(\{c\})$,

induced by a min-semi-balanced system $\mathcal{S} := \{\, \{a, c\}, \{b, c\}, \{c\} \,\}$ on N.

Purely semi-balanced systems and the exceptional set

Once $|N| \geq 3$, the superfluous inequalities for the description of $\mathcal{E}(N)$ are, surprisingly, all those which correspond to balanced systems on N.

Definition (purely semi-balanced set system)

A semi-balanced set system $\mathcal{S}$ (on N) which is not balanced (on N) will be called *purely semi-balanced* (on N).

In case of a purely semi-balanced system we have a special terminology.

Definition (exceptional set)

Given a non-trivial set system $\mathcal{S}$ on N, we will say that a set $T \in \mathcal{S}$ is *exceptional* within $\mathcal{S}$ if there exists a linear combination $\sum_{S \in \mathcal{S}} \lambda_S \cdot \chi_S$ yielding a constant vector in $\mathbb{R}^N$ with $\lambda_T < 0$ and $\lambda_S \geq 0$ for $S \in \mathcal{S} \setminus \{T\}$.

The exceptional set is **unique** in case of a min-semi-balanced system.

Re-writings for a min-semi-balanced inequality (picture)

Important fact is that, in case of a min-semi-balanced set system $\mathcal{S}$ on N, the coefficients λ_S in the unique affine semi-conic combination

$$\sum_{S\in\mathcal{S}} \lambda_S \cdot \chi_S = r \cdot \chi_N\,, \quad \lambda_S, r \in \mathbb{R}, \quad \sum_{S\in\mathcal{S}} \lambda_S = 1\,,$$

are **rational** numbers!

Hence, it can be *standardized* by multiplying by a natural number ℓ so that $\boxed{\alpha_S := \ell \cdot \lambda_S}$, $S \in \mathcal{S}$, become integers with no common prime divisor:

$$\sum_{S\in\mathcal{S}} \alpha_S \cdot \chi_S = \alpha_N \cdot \chi_N\,,$$ where α_S are integers and so is $\alpha_N := \ell \cdot r$.

One can additionally put $\alpha_\emptyset := -\alpha_N + \sum_{S\in\mathcal{S}} \alpha_S \in \mathbb{Z}^+$ and get

$$\sum_{S\in\mathcal{S}} \alpha_S \cdot \chi_S = \alpha_N \cdot \chi_N + \alpha_\emptyset \cdot \chi_\emptyset\,,$$ where all the coefficients α_S are integers.

Pictorial representation of a min-semi-balanced system

Provided there is (an exceptional set) $T \in \mathcal{S}$ with $\lambda_T < 0$, that is, $\alpha_T < 0$, one can re-write that in the form

$$\sum_{S \in \mathcal{S} \setminus \{T\}} \alpha_S \cdot \chi_S = \alpha_N \cdot \chi_N + (-\alpha_T) \cdot \chi_T + \alpha_\emptyset \cdot \chi_\emptyset$$

with *non-negative integers* as coefficients.

A diagram representing a set system $\mathcal{S}$ has the form of a pair of two-dimensional arrays whose entries are colorful boxes;

the arrays encode the sides of the above vector equality.

The rows of these arrays correspond to the elements of the basic set N.

Each set has a specific color and the values α_S are expressed by repeating columns (of the same color).

The grand coalition has reserved the black color, a possible exceptional set is in grey, and the empty set is in white (= a blank column).

Example: a diagram representing a set system/inequality

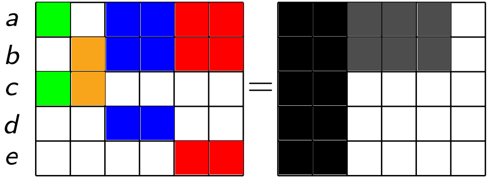
Example (pictorial representation of an inequality)

Take $N := \{a, b, c, d, e\}$ and put $\mathcal{S} := \{\, ab, ac, bc, abd, abe \,\}$. Then

$$1 \cdot \chi_{ac} + 1 \cdot \chi_{bc} + 2 \cdot \chi_{abd} + 2 \cdot \chi_{abe} = 2 \cdot \chi_{N} + 3 \cdot \chi_{ab} + 1 \cdot \chi_{\emptyset}$$

allows one to observe that $\mathcal{S}$ is semi-balanced on N. Because the vectors $\{\chi_S \,:\, S \in \mathcal{S}\}$ are linearly independent, $\mathcal{S}$ is minimal (this is a sufficient condition for minimality). The respective inequality is

$$m(ac) + m(bc) + 2 \cdot m(abd) + 2 \cdot m(abe) \,\leq\, 2 \cdot m(N) + 3 \cdot m(ab) + m(\emptyset)$$



Towards minimal inequality description of exact games

The observation that min-balanced set systems on N yield superfluous inequalities has already been done by Lohmann *et al.* in 2012.

E. Lohmann, P. Borm, P.J.-J. Herings. Minimal exact balancedness. *Mathematical Social Sciences*, 64:127–135, 2012.

The inequalities reported by them were also associated with particular set systems. (although in a slightly technically different way we did this here)

They also noticed that even a purely min-semi-balanced system can induce a superfluous inequality.

In our 2022 paper we succeeded to characterize those min-semi-balanced set systems which yield *facet-defining inequalities*. We introduced an appropriate concept of a *combinatorial decomposition* of a min-semi-balanced set system which leads to the solution.

Indecomposable min-semi-balanced systems

Definition (decomposition, indecomposable system)

Assume $|N| \geq 3$. Given a purely min-semi-balanced set system $\mathcal{S}$ on N and $E \subseteq N$ with $E \notin (\mathcal{S} \cup \{\emptyset, N\})$ we say that E yields a *decomposition* of $\mathcal{S}$ if E is an exceptional set within $\mathcal{S} \cup \{E\}$.

A purely min-semi-balanced set system on N will be called *indecomposable* if it has no decomposition.

The point of the above defined combinatorial concept of a decomposition is that (in)decomposability is testable by a simple algorithm.
(details are omitted here)

Theorem (our main result)

If $|N| \geq 3$, then an inequality for $m \in \mathbb{R}^{\mathcal{P}(N)}$ is facet-defining for $\mathcal{E}(N)$ iff its multiple is given by an indecomposable min-semi-balanced system on N.

Hidden symmetry among inequalities and other remarks

Basic observation here is that the *anti-dual transformation* $m \mapsto m^{\diamond}$, where $\boxed{m^{\diamond}(S) := m(N \setminus S) - m(N) \text{ for } S \subseteq N}$, maps $\mathcal{E}(N)$ onto itself.

This allows one to show that, if some inequality is facet-defining for $\mathcal{E}(N)$ then its *conjugate inequality* is also facet-defining: one replaces any set by its complement ($S \leftrightarrow N \setminus S$) while the coefficients remain unchanged. (the same holds for $\mathcal{B}(N)$ in place of $\mathcal{E}(N)$)

On the level of set systems this also means replacing a set by its complement, giving the so-called *complementary system*. The conjugate inequalities are easily recognizable on basis their pictorial diagrams.

Additionally, there is a certain one-to-many correspondence between min-balanced systems $\mathcal{B}$ on N involving at least 3 sets and *purely* min-semi-balanced set systems $\mathcal{S}$ on N: a set $S \in \mathcal{B}$ is chosen and replaced by its complement. The point is that every purely min-semi-balanced system on N can be obtained in this way !

Our web catalogue of inequalities for $|N| \leq 6$

Since the criterion of *indecomposability* can easily be implemented, there is a combinatorial way to obtain/enumerate all facet-defining inequalities for the cone of exact games for general N.
(once one knows all min-semi-balanced systems over that basic set N)

These are the numbers of facets of the exact cone for $n = |N| \leq 6$:

Number of players	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
Number of facets	1	6	44	280	7006
Number of its permutational types	1	2	6	16	53

Our catalogue is available on the web page

http://gogo.utia.cas.cz/indecomposable-min-semi-balanced-catalogue/ .

The inequalities are represented by our pictorial diagrams in the catalogue.

Alternative way: extreme exact games

An alternative way to describe a polyhedral cone (= instead of finding its facets) is to enumerate the extreme rays of its pointed version.

Applying this in the context of the cone of *non-negative exact games* leads to the super-exponential growth of the number of these rays with $|N|$, which was confirmed by computing *extreme coherent lower probabilities*.

E. Quaeghebeur, G. de Cooman. Extreme lower probabilities. *Fuzzy Sets and Systems*, 159:2163–2175, 2008.

M. Studený, V. Kratochvíl. Linear criterion for testing the extremity of an exact game based on its finest min-representation. *International Journal of Approximate Reasoning*, 101:49–68, 2018.

Thus, it looks like there is no chance to characterize exact games, or coherent lower probabilities, in these terms.

In our 2018 paper, we offered a criterion to recognize whether a given standardized exact game is extreme or not.

The cone of supermodular/convex games

Definition (supermodular game)

We say that a game $m : \mathcal{P}(N) \to \mathbb{R}$ is *supermodular* if it satisfies

$$m(S \cup T) + m(S \cap T) \; \geq \; m(S) + m(T) \qquad \text{for any } S, T \subseteq N\,.$$

Traditional terminology in game theory is, however, a *convex game*.

In the context of imprecise probabilities, it corresponds to the concept of a *2-monotone lower probability*.

E. Miranda, I. Montes. Shapley and Banzhaf values as probability transformations. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 26(6):917–947, 2018.

S. Destercke. Independence and 2-mononicity: nice to have, hard to keep? *International Journal of Approximate Reasoning*, 54(4):478–490, 2013.

Inequality description of the cone of supermodular games

Facet-defining inequalities of this cone are well-known. These are

$$m(\{i,j\} \cup L) + m(L) - m(\{i\} \cup L) - m(\{j\} \cup L) \geq 0$$

for $m \in \mathbb{R}^{\mathcal{P}(N)}$, where $L \subset N$ and $i,j \in N \setminus L$ are distinct.

On the other hand, the characterization of extreme rays of its pointed version is difficult for the same reason as in case of exact games.

In our 2016 paper with Tomáš Kroupa, we offered a criterion to recognize whether a given standardized supermodular game is extreme or not.

J. Kuipers, D. Vermeulen, M. Voorneveld.
A generalization of the Shapley-Ichiishi result.
International Journal of Game Theory, 39:585–602, 2010.

M. Studený, T. Kroupa. Core-based criterion for extreme supermodular functions. *Discrete Applied Mathematics*, 206:122–151, 2016.

The cone of totally balanced games

A *subgame* of a game m over N specified by a subset $M \subseteq N$, $|M| \geq 2$, is a game over M which has as its characteristic function the restriction of m on the power set $\mathcal{P}(M)$ of M.

It is known that a subgame of a balanced game need not be balanced.

Definition (totally balanced game)

We say that a game $m : \mathcal{P}(N) \to \mathbb{R}$ is *totally balanced* if every its subgame is balanced.

This class of games, whose counterparts were probably not studied in context of imprecise probabilities, was a topic of study in game theory. The inclusions $\mathcal{E}(N) \subset \mathcal{T}(N) \subset \mathcal{B}(N)$ for $|N| \geq 3$ are well-known.

The task of characterization of facets of this cone $\mathcal{T}(N)$ was solved.

T. Kroupa, M. Studený. Facets of the cone of totally balanced games. *Mathematical Methods of Operations Research*, 90(2):271–300, 2019.