Bringing philosophy and IP into conversation: Puzzles and challenges for decision theory

Jason Konek

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Department of Philosophy, Bristol
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Outline

- Desirability and Choice without Act-State Independence
- The Value of Incomparability

Desirability and Choice without Act-State Independence

Coherent sets of desirable gambles

Let Ω be a finite possibility space.

A gamble $g : \Omega \to \mathbb{R}$ is an uncertain reward. We collect them in $\mathcal{L}(\Omega)$.

A set of gambles $\mathcal{D} \subseteq \mathcal{L}(\Omega)$ that You judge desirable (prefer to the status quo, 0), is **coherent** iff

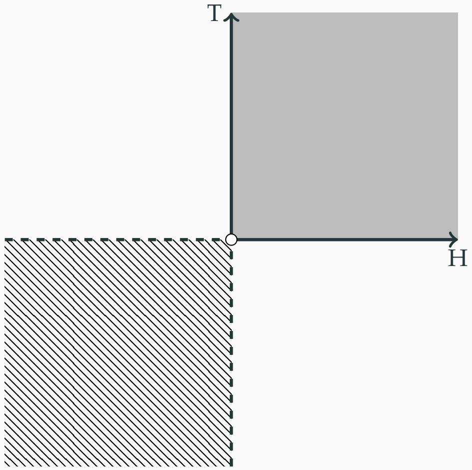
D1. $0 \notin \mathcal{D}$

D2. If $f \geqslant 0$ and $f \neq 0$, then $f \in \mathcal{D}$

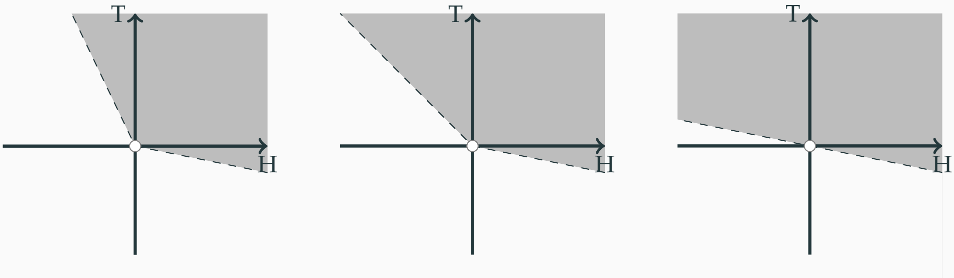
D3. If $f \in \mathcal{D}$ and $\lambda > 0$ then $\lambda f \in \mathcal{D}$

D4. If $f, g \in \mathcal{D}$ then $f + g \in \mathcal{D}$

Coherent sets of desirable gambles

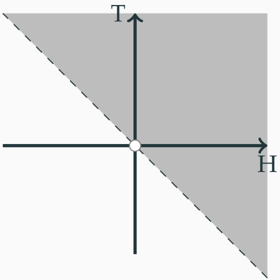


Coherent sets of desirable gambles

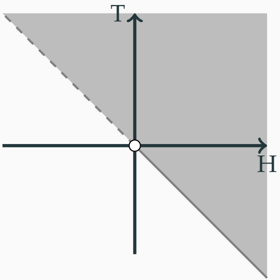


Rather imprecise (non-committal) | More precise: (larger set, more committal) | Open half-space: precise prob model

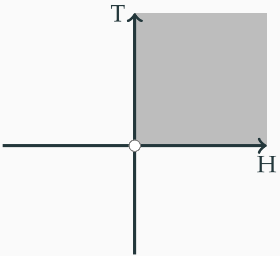
Coherent sets of desirable gambles



Symmetric half-space:
Uniform probability



Half-open boundary:
Infinitesimally biased



Smallest coherent $\mathcal{D}$
"The vacuous model"

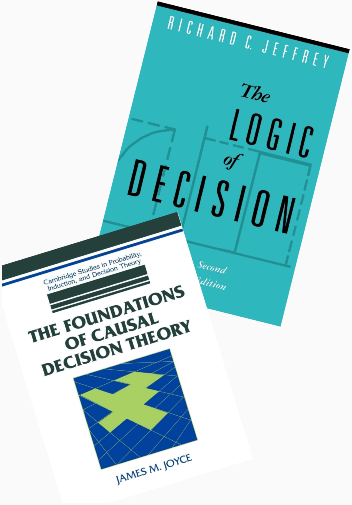
Example: Protection

Example: Protection

		
f : **Buy Protection**	-410	-10
g : **Don't Buy Protection**	-400	0

$g - f \in \mathcal{D}$ **(by D2)**

Act-State Independence



- Savage: No trouble with Dominance. The decision problem is ill-posed. Probabilities of states must be independent of which gamble is chosen.
- **Problem 1.** Independence: evidential or causal?
- **Problem 2.** It's the **decision-maker's views about independence that matter**.
  - No "choice events" in the desirability framework
  - No way to model how info about choices affects decision-maker's views about states

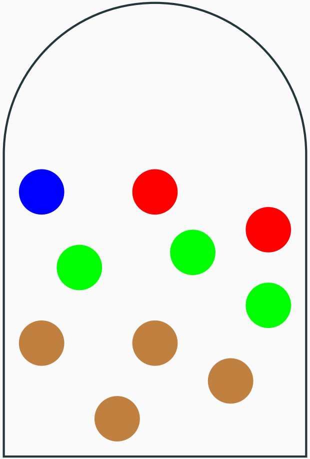
Evidential Decision Theory

- $\succ$ is a relation on a σ-algrebra $\mathcal{F}$ of Ω
- $\succ$ is an asymmetric weak order (nontriviality, nullity)
- **Continuity**: If $A_n \uparrow A$ or $A_n \downarrow A$ and $B \succ A \succ C$, then $B \succ A_n \succ C$
- **Averaging**: If A and B are disjoint then
  - $A \succ B$ implies $A \succ A \cup B \succ B$
  - $A \sim B$ ($A \not\succ B$ and $B \not\succ A$) implies $A \sim A \cup B \sim B$
- **Impartiality**: If A, B, C are mutually disjoint and

$$A \sim B\ ,\ A \succ C \text{ or } C \succ A\ ,\ A \cup C \sim B \cup C$$

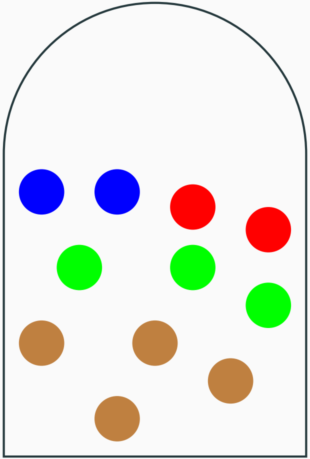
then $A \cup D \sim B \cup D$ whenever A, B, D are mutually disjoint.

Impartiality: An Example



- Blue or Red: £0. Green or Brown: £100.
- Blue $\sim$ Red
- Green $\succ$ Blue $\sim$ Red, Brown $\succ$ Blue $\sim$ Red
- Blue or Green $\succ$ Red or Green
    - Learning Blue or Green is better news than learning Red or Green

Impartiality: An Example



- Blue or Red: £0. Green or Brown: £100.
- Blue $\sim$ Red
- Green $\succ$ Blue $\sim$ Red, Brown $\succ$ Blue $\sim$ Red
- Blue or Green $\sim$ Red or Green
  - Learning Blue or Green, on the one hand, or Red or Green on the other—equally good news
  - Happens iff Blue and Red are equiprobable
- Blue or Brown $\sim$ Red or Brown

Evidential Decision Theory

Jeffrey's Representation Theorem

If $\succ$ satisfies the Jeffrey-Bolker axioms, then there is a probability measure P and signed measure M on $\mathcal{F}$ with $M \gg P$ such that

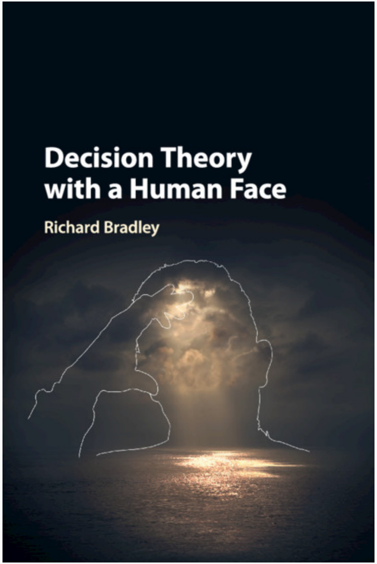
$$A \succ B \textit{ iff } \frac{M(A)}{P(A)} > \frac{M(B)}{P(B)}$$

Letting $U = M/P$ and noting $M(A) = \int_A u \, \mathrm{d}P$ yields

$$A \succ B \textit{ iff } U(A) > U(B) \textit{ iff } \frac{\int_A u \, \mathrm{d}P}{P(A)} > \frac{\int_B u \, \mathrm{d}P}{P(B)}$$

Moreover U is unique up to fractional linear transformation.

Evidential Decision Theory



- Jeffrey treats $A \not\succ B$ and $B \not\succ A$ as indifference
  - Completeness
- Bradley provides IP version of EDT
  - Bradley's RT: $\succeq$ full agrees with a set of $< P, M >$ pairs.
  - Bradley's axiomatization assumes $\succeq$ has a mininimal (preserves strict part) Jeffrey coherent extension $\succeq'$
- Open questions
  - Satisfactory axiomatization of coherent partial evidential preference relations
  - Choice functions representable by sets of coherent partial evidential preference relations

EDT and Desirability

Let $\mathcal{W}$ be a finite possibility space and $\mathcal{L}(\mathcal{W})$ be the set of gambles on Ω.

$\mathcal{G} \subseteq \mathcal{L}(\mathcal{W})$ represents the gambles that are available to You.

Let $\Omega = \mathcal{W} \times \mathcal{G}$ and $\mathcal{A}$ be the set of act events $E_g = \{< \omega, f >\in \Omega \mid f = g\}$.

Let $\mathcal{L}(\Omega)$ be the set of bounded gambles on Ω.

Let $\rho_g(\omega, f) = g(\omega)$ if $f = g$, 0 otherwise.

Let $\mathcal{D} \subseteq \mathcal{L}(\Omega)$ be a coherent set of desirable gambles

Interval Dominance: $f > g$ iff

- $\sup\left\{\epsilon > 0 \mid \rho_f - \epsilon \in \mathcal{D}_{E_f}\right\} > \inf\left\{\epsilon > 0 \mid \epsilon - \rho_g \in \mathcal{D}_{E_g}\right\}$
- OR $\sup\left\{\epsilon > 0 \mid \rho_f - \epsilon \in \mathcal{D}_{E_f}\right\} = \inf\left\{\epsilon > 0 \mid \epsilon - \rho_g \in \mathcal{D}_{E_g}\right\}$ and $\neg\left[\rho_f - \sup\left\{\epsilon > 0 \mid \rho_f - \epsilon \in \mathcal{D}_{E_f}\right\} \in \mathcal{D}_{E_f} \textit{ iff } \inf\left\{\epsilon > 0 \mid \epsilon - \rho_g \in \mathcal{D}_{E_g}\right\} - \rho_g \in \mathcal{D}_{E_g}\right]$

Example: Protection

	 ω_1	 ω_2
f : Buy Protection	-410	-10
g : Don't Buy Protection	-400	0

- $\mathcal{G} = \{f, g\}$
- $\Omega = \{< \omega_1, f >, < \omega_2, f >, < \omega_1, g >, < \omega_2, g >\}$
- $\mathcal{D}$ is the coherent set of desirable gambles based on $\mathcal{M} = \left\{ m =< m_1, m_2, m_3, m_4 > \mid \frac{m_2}{m_1+m_2} \geqslant \frac{9}{10}, \frac{m_3}{m_3+m_4} \geqslant \frac{9}{10} \right\}$
- $\sup \left\{ \epsilon > 0 \mid \rho_f - \epsilon \in \mathcal{D}_{E_f} \right\} = -50 > -360 = \inf \left\{ \epsilon > 0 \mid \epsilon - \rho_g \in \mathcal{D}_{E_g} \right\} \Rightarrow f > g$

EDT and Partial Preferences on Gambles

Let $\mathcal{W}$ be a finite possibility space and $\mathcal{L}(\mathcal{W})$ be the set of gambles on Ω.

$\mathcal{G} \subseteq \mathcal{L}(\mathcal{W})$ represents the gambles that are available to You.

Let $\Omega = \mathcal{W} \times \mathcal{G}$ and $\mathcal{A}$ be the set of act events $E_g = \{< \omega, f >\in \Omega \mid f = g\}$.

Let $\mathcal{L}(\Omega)$ be the set of bounded gambles on Ω.

Let $\rho_g(\omega, f) = g(\omega)$ if $f = g$, 0 otherwise.

Let $\mathcal{L}^+$ be the linear space of partial gambles generated by $\mathcal{A}$ (*i.e.*, $\gamma \mid A$ in $\mathcal{L}^+$ for all $\gamma \in \mathcal{L}(\Omega)$, $A \in \mathcal{A}$; $\mathcal{L}^+$ closed under function addition and scaling).

$$f \succ g \text{ iff } \rho_f \mid E_f \succ \rho_g \mid E_g.$$

Open Question: Axiomatize coherent partial preference relations on $\mathcal{L}^+$ (Suppes and Zanotti 1982).

EDT and Probability Filters

- $\mathcal{L}(\Omega)$ is the set of bounded gambles on $\Omega = \mathcal{W} \times \mathcal{G}$.
- $\mathcal{P}$ is the set of coherent linear previsions on $\mathcal{L}(\Omega)$ and $\mathfrak{P}(\mathcal{P})$ is the power set of $\mathcal{P}$
- $\mathcal{F} \subseteq \mathfrak{P}(\mathcal{P})$ is coherent iff it is a proper filter:
  - F1. $\mathcal{F} \neq \emptyset$
  - F2. $P, Q \in \mathcal{F}$ implies $P \cap Q \in \mathcal{F}$
  - F3. $P \in \mathcal{F}$ and $P \subseteq Q$ implies $Q \in \mathcal{F}$
  - F4. $\emptyset \notin \mathcal{F}$
- **Rejection**: $R : \mathrm{Fin}(\mathcal{G}) \to \mathrm{Fin}(\mathcal{G})$
- **E-Admissibility**: $g \in R(A)$ iff $\left\{p \mid (\exists f \in A) p(\rho_f \mid E_f) > p(\rho_g \mid E_g)\right\} \in \mathcal{F}$
- **Maximality**: $g \in R(A)$ iff $(\exists f \in A) \left\{p \mid p(\rho_f \mid E_f) > p(\rho_g \mid E_g)\right\} \in \mathcal{F}$

CDT on Protection

		
f : **Buy Protection**	-410	-10
g : **Don't Buy Protection**	-400	0

Choosing *f* **causes** your windshield to remain unsmashed

Causal vs. Evidential Decision Theory: Twin Prisoner's Dilemma

		
Rat	-10	0
Silence	-20	-1

EDT: Silent, CDT: Rat

$\succ$ is a relation on $\mathfrak{P}(\Omega) \times \mathcal{C}$ where $\mathcal{C} \subseteq \mathfrak{P}(\Omega)$ is a set of "conditions" which include the available acts

Joyce's Representation Theorem

If $\succ$ satisfies Joyce's axioms, then there is a unique supposition function $P : \mathfrak{P}(\Omega) \times \mathcal{C} \to [0, 1]$ and utility function $u : \Omega \to \mathbb{R}$ such that $A \succ B$ iff

$$U(A) = \sum_{\omega \in \Omega} P(\{\omega\} \parallel A) u(\omega) > \sum_{\omega \in \Omega} P(\{\omega\} \parallel A) u(\omega) = U(B)$$

Moreover u is unique up to positive affine transformation.

- $\sum_{\omega \in \Omega} P(\{\omega\} \,\|\, A) u(\omega)$ can be written as

$$\sum_{\omega \in \Omega} P(\{\omega\} \,\|\, \Omega) \sum_{\omega' \in \Omega} would_A(\omega, \omega') u(\omega')$$

- $would_A$ is an **imaging function**
- $would_A(\omega, \cdot)$ is a pmf that puts all probability mass on A
- $would_A(\omega, \omega')$ is roughly the probability, at ω, that ω' would result ***were you to make A true***

CDT and Desirability:

- Ω is a finite possibility space
- $u : \Omega \to \mathbb{R}$ is a linear utility
- $would_A$ is an imaging function
- $\mathcal{D} \subseteq \mathcal{L}(\Omega)$ is a coherent set of desirable gambles
- $A \succ B$ iff $would_A \cdot u - would_B \cdot u \in \mathcal{D}$

The Value of Incomparability

Guidance Value of Choice Functions

Let Ω be a finite possibility space.

Let $\mathrm{Fin}(\mathcal{L}(\Omega))$ be the space of all finite subsets of $\mathcal{L}(\Omega)$

Let $C : \mathrm{Fin}(\mathcal{L}(\Omega)) \to \mathrm{Fin}(\mathcal{L}(\Omega))$ be a choice function (so $C(A) \subseteq A$)

Let μ be a measure on $\mathrm{Fin}(\mathcal{L}(\Omega))$

Question: If we now know ω and μ, but not exactly which decision problems C was used to address, can we evaluate how well C did at guiding choice? (*cf.* Schervish [1989])

Pettigrew's Approach

- Guidance value in the face of incomparability
  - Pettigrew: Decision-makers randomize over their choice set
- If $\mathcal{C}(A) = \{g_1, \ldots, g_n\}$ then $p(A, C(A)) = \langle p_1, \ldots, p_n \rangle$ is a pmf.
- Let $u_\omega(\mathcal{C}(A)) = \langle g_1(\omega), \ldots, g_n(\omega) \rangle$
- Let $\mathcal{E}_\omega(A, \mathcal{C}(A)) = p(A, C(A)) \cdot u_\omega(\mathcal{C}(A))$
  - Expected payout at ω if randomly picking from $\mathcal{C}(A)$ via $p(A, C(A))$
- **Guidance Value**:

$$g_\omega(\mathcal{C}) = \int_{\mathrm{Fin}(\mathcal{L}(\Omega))} \mathcal{E}_\omega(A, \mathcal{C}(A))\, \mathrm{d}\mu$$

Pettigrew's Challenge

1. No reasonable choice function is dominated in terms of guidance value (guaranteed to be worse than some other choice function)
2. Every imprecise C is dominated

C. Every imprecise C is unreasonable

Strict $\mathcal{C}$-Propriety

Suppose that for all $i \leqslant n$

$$g_\omega(\mathcal{C}) = \int_{\mathrm{Fin}(\mathcal{L}(\Omega))} \mathcal{E}_\omega(A, \mathcal{C}(A))\,\mathrm{d}\mu$$

Suppose further that $\mu(X) > 0$ for any non-degenerate $X \subseteq \mathrm{Fin}(\mathcal{L}(\Omega))$.

Then for any probability mass function $p : \Omega \to \mathbb{R}$ and any $\mathcal{C} \neq \mathcal{C}_p$

$$p \cdot g(\mathcal{C}) < p \cdot g(\mathcal{C}_p)$$

unless $\mathcal{C} \neq \mathcal{C}_p$ on a set of measure zero.

g is a **strictly $\mathcal{C}$-proper**.

Definition

C is **Bayes optimal relative to** g if and only if C maximizes expected guidance value relative to some pmf p.

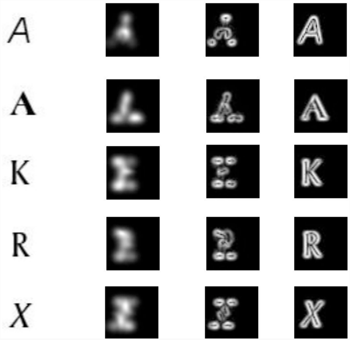
Definition

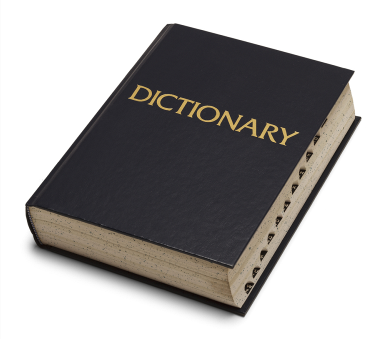
C is **admissible relative to** g if and only if there is no C' such that $g(C) < g(C')$.

Wald's Complete Class Theorem

Under mild conditions, satisfied by strictly C-proper g, C is Bayes optimal (relative to g) if and only if C is admissible (relative to g).

Resolving Decision Problems with Incomparability





- De Bock & de Cooman 2014: If multiple candidate words are non-rejected, cross-check against a dictionary
- **Proponents of IP must think systematically about how to resolve incomparability!**

References

L. J. Savage. Elicitation of personal probabilities and expectations. Journal of the American Statistical Association, 66:783–801, 1971.

Mark Schervish. A general method for comparing probability assessors. The Annals of Statistics, 17(4):1856–1879, 1989.