Coalitional game theory VS Imprecise probabilities: Two sides of the same coin

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My collaborations | **Other authors**

Coalitional games


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L.Shapley


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Distortions models


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A.Bronevich


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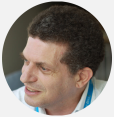
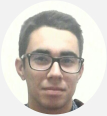
L.Utkin


T.Seidenfeld

Approximations of lower probabilities


P.Vicig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D.Skulj

Overview

Lower probabilities

Coalitional games

Game solutions - Centroids

Lower previsions VS coalitional games

Conclusions

Lower probabilities

A *lower probability* is a function $\underline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$ satisfying

- Monotonicity: $\underline{P}(A) \leq \underline{P}(B)$ if $A \subseteq B$.
- Normalisation: $\underline{P}(\emptyset) = 0$, $\underline{P}(\mathcal{X}) = 1$.

Its conjugate *upper probability* $\overline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$ is defined by:

$$\overline{P}(A) = 1 - \underline{P}(A^c) \quad \forall A \subseteq \mathcal{X}.$$

Interpretations:

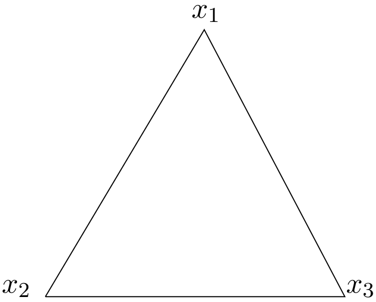
- *Behavioural:* supremum buying ($\underline{P}(A)$) or infimum selling ($\overline{P}(A)$) prices a bet on A.
- *Epistemic:* $[\underline{P}(A), \overline{P}(A)]$ gives bounds for the unkown value of $P_0(A)$.
- *Coalitional games.*

Credal set

The *credal set* associated with a lower probability is given by:

$$\mathcal{M}(\underline{P}) = \{P \text{ probability measures} \mid P(A) \geq \underline{P}(A)\ \forall A \subseteq \mathcal{X}\}.$$

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}$	0.1	0.3	0.2	0.5	0.3	0.6
$\overline{P}$	0.4	0.7	0.5	0.8	0.7	0.9



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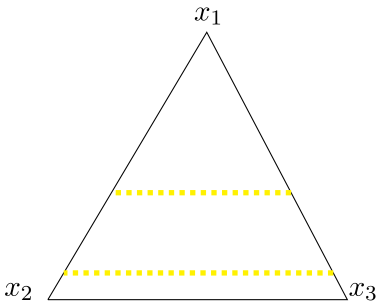

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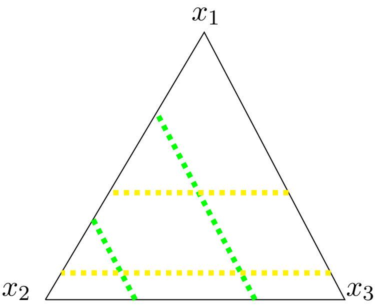

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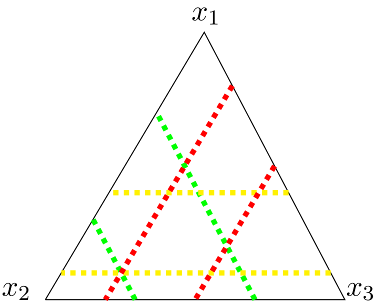

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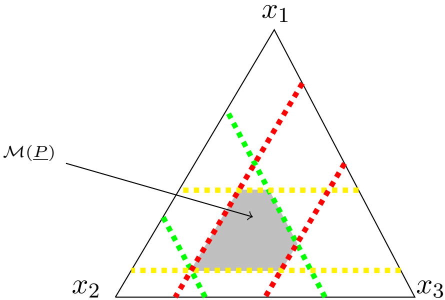

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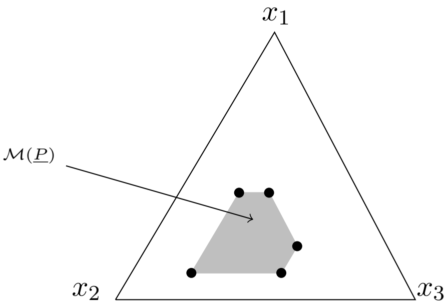

Rationality conditions

A lower probability . . .

. . . *avoids sure loss* if $\mathcal{M}(\underline{P}) \neq \emptyset$.

. . . *is coherent* if $\underline{P}(A) = \min\{P(A) \mid P \in \mathcal{M}(\underline{P})\}\ \forall A \subseteq \mathcal{X}$.

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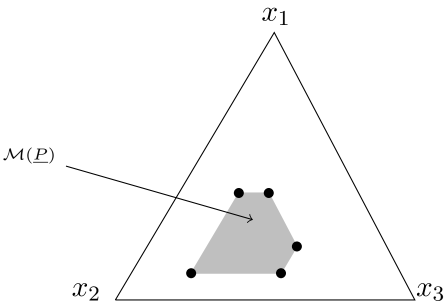

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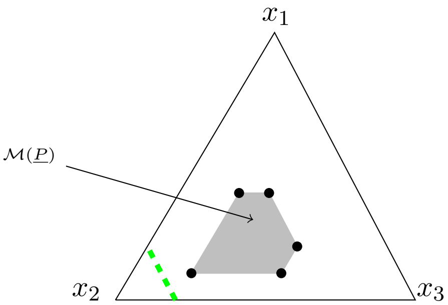

2-monotonicity

A lower probability $\underline{P}$ is *2-monotone* if:

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B) \quad \forall A, B \subseteq \mathcal{X}.$$

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Properties:

- $\underline{P}$ has a unique extension of gambles $\to$ Choquet integral.
- Characterisation through the Möbius inverse.
- Conditioning.
- Formula for the extreme points.

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Independence and 2-monotonicity: Nice to have, hard to keep☆

CrossMark

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Example: distortion models

The *linear vacuous* model (LV) induced by the probability measure P_0 and $\delta \in (0,1)$ is defined by:

$$\underline{P}_{LV}(A) = (1-\delta)P_0(A) \quad \forall A \subset \mathcal{X}.$$

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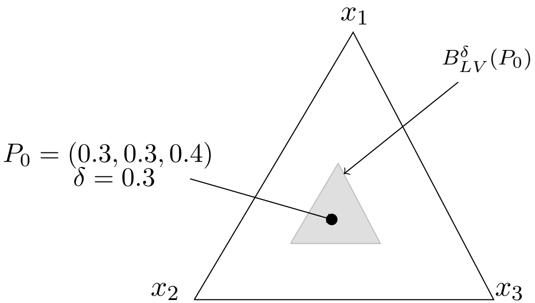

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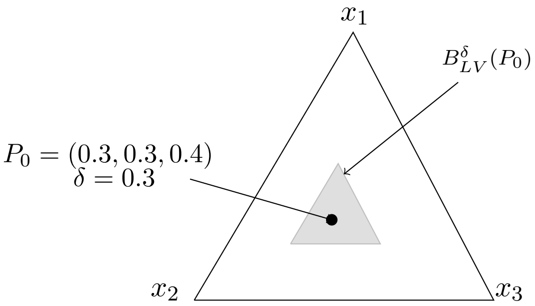
The *Pari Mutuel* model (PMM) induced by the probability measure P_0 and $\delta > 0$ is defined by:

$$\underline{P}_{PMM}(A) = \max\{(1 + \delta)P_0(A) - \delta, 0\} \quad \forall A \subseteq \mathcal{X}.$$



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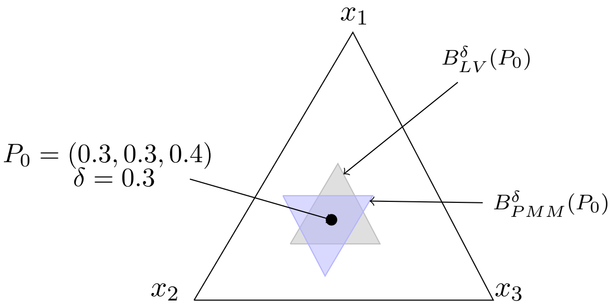

Overview

Coalitional games

Basic concepts:

- Set of players: $\mathcal{X} = \{x_1, \ldots, x_n\}$.
- Coalition of players: $A \subseteq \mathcal{X}$.
- Game: $\nu : \mathcal{P}(\mathcal{X}) \to [0, \infty)$.
- $\nu(A)$: minimum reward guaranteed by coalition A.
- $\nu(\mathcal{X})$: total reward.

We impose the following conditions:

- $\nu(\emptyset) = 0$ and $\nu(\mathcal{X}) = 1$.
- $\nu(A) \leq \nu(B)$ when $A \subseteq B$.

Coalitional games VS Lower probabilities

Coalitional game theory	**Lower probabilities**
Normalised game (ν)	Lower probability ($\underline{P}$)
Conjugate ($\bar{\nu}$)	Conjugate upper probability ($\overline{P}$)
Balanced game	Lower probability avoiding sure loss
Exact game	Coherent lower probability
Core (core(ν))	Credal set ($\mathcal{M}(\underline{P})$)
Convex game	2-monotone lower probability

Coalitional games VS Lower probabilities: Two sides of the same coin...

Extreme points

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Extreme points of coherent probabilities in finite spaces

Anton Wallner

Theorem 5.13: The number of extreme points of the credal set associated with a coherent lower probability is $|\mathcal{X}|!$.

Chapters in Game Theory pp 83–97 | Cite as

Springer Link

On the Number of Extreme Points of the Core of a Transferable Utility Game

Jean Derks & Jeroen Kuipers

Chapter

Part of the Theory and Decision Library C: book series (TDLC,volume 31)

Corollary 4.6: The core of a n-person exact game has at most $n!$ extreme points.

Overview

Lower probabilities

Coalitional games

Game solutions - Centroids

Lower previsions VS coalitional games

Conclusions

Game solutions - Centroids

Solution of the game: a "fair" way of dividing the total reward.
Centroid: a representative of the credal set.

1. Shapley value
2. Average of the extreme points
3. Incenter
4. Contraction centroid

Technical assumptions:

- $\mathcal{X}$ is finite.
- $\underline{P}(A) \in (0, 1)$ if $A \neq \emptyset, \mathcal{X}$.

1.Shapley value

$$\Phi_1^{\underline{P}}(\{x\}) = \sum_{A \mid x \notin A} \frac{|A|!(n-|A|-1)!}{n!}\big(\underline{P}(A \cup \{x\}) - \underline{P}(A)\big).$$

Properties:

- If $\underline{P}$ is 2-monotone, $\Phi_1^{\underline{P}} \in \mathcal{M}(\underline{P})$.
- If $\underline{P}$ is not 2-monotone, $\Phi_1^{\underline{P}}$ may not belong to $\mathcal{M}(\underline{P})$.
- Axiomatic definition.

A VALUE FOR n-PERSON GAMES

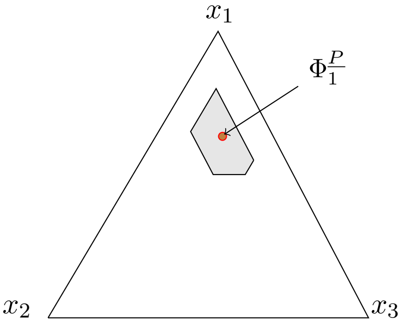
L. S. Shapley

P-295

18 March 1952

1.Shapley value: example

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}(A)$	0.5	0.1	0.1	0.65	0.75	0.2



2.Average of the extreme points

If $P_1, \ldots, P_k$ are the extreme points of $\mathcal{M}(\underline{P})$, the *average of the extreme points is* given by:

$$\Phi_2^{\underline{P}}(\{x\}) = \frac{1}{k}\sum_{i=1}^{k} P_i(\{x\}).$$

Properties:

- $\Phi_2^{\underline{P}} \in \mathcal{M}(\underline{P})$.
- If $\underline{P}$ is 2-monotone and $\mathcal{M}(\underline{P})$ has $n!$ different extreme points, $\Phi_1^{\underline{P}} = \Phi_2^{\underline{P}}$.

2.Average of the extreme points: example

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
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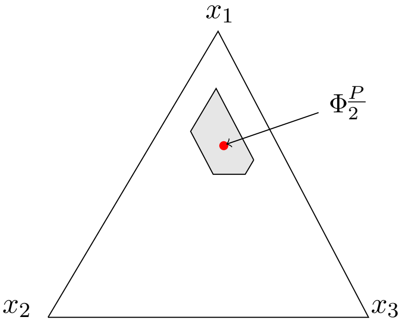

3.Incenter (w.r.t. TV-distance)

TV-distance:

$$d_{TV}(P,Q) = \max_{A \subseteq \mathcal{X}} |P(A) - Q(A)|.$$

Incenter radius:

$$\alpha_I = \sup \left\{ \alpha \mid \exists P_0 \in \mathcal{M}(\underline{P}) \text{ satisfying } B_o^{\alpha}(P_0) \subseteq \mathcal{M}(\underline{P}) \right\}.$$

Incenter: any $\Phi_3^{\underline{P}}$ satisfying $B_o^{\alpha_I}\left(\Phi_3^{\underline{P}}\right) \subseteq \mathcal{M}(\underline{P})$.

Properties:

- An incenter always exists *(whenever $\mathcal{M}(\underline{P})$ has a non-empty interior)*.
- The incenter may not be unique.
- Any incenter belongs to $\mathcal{M}(\underline{P})$.

3.Incenter: example

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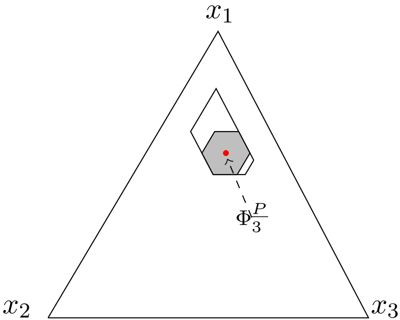

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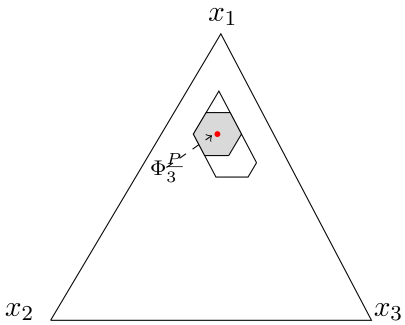

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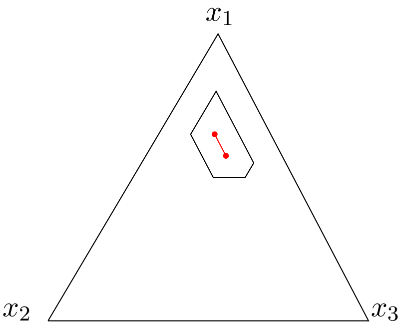
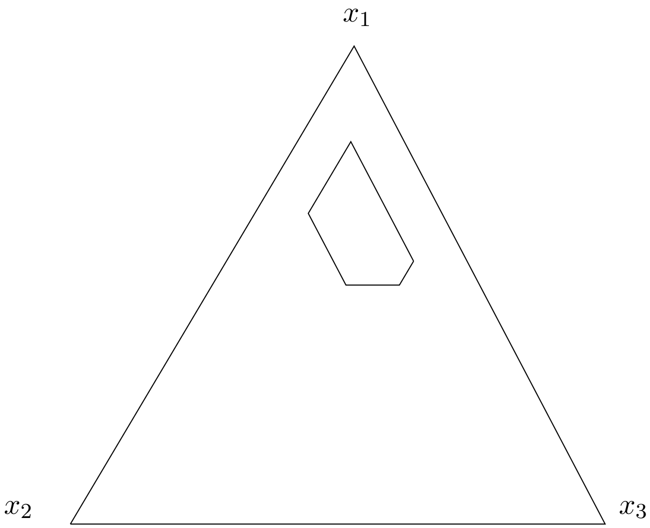


4.Contraction centroid: motivation



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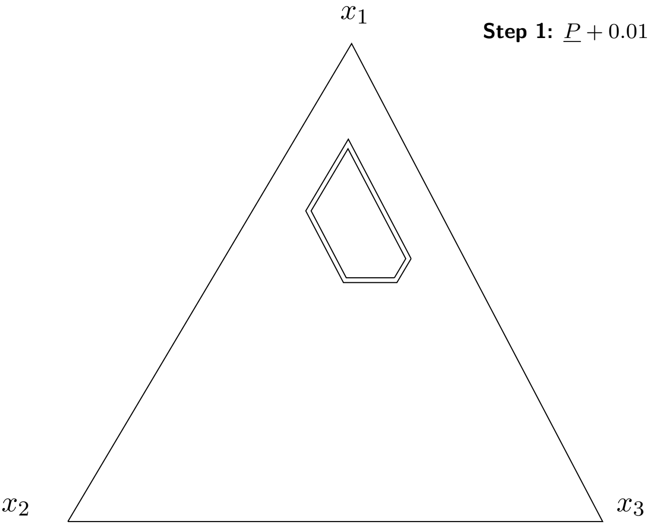
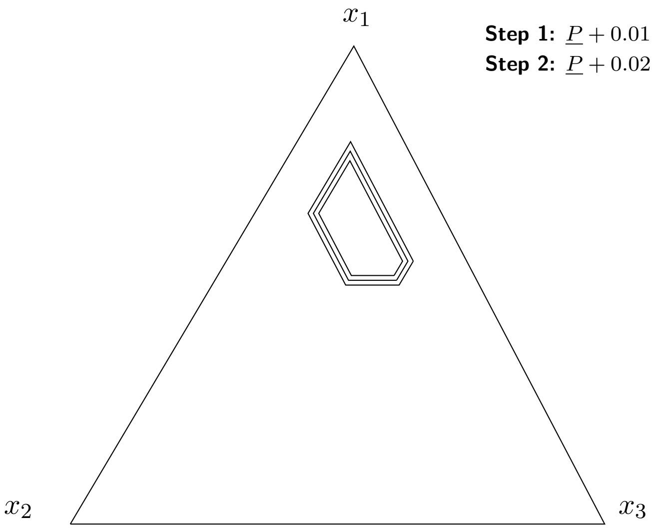


Step 1: $\underline{P}+0.01$, $\overline{P}-0.01$

4.Contraction centroid: motivation

Step 1: $\underline{P} + 0.01$, $\overline{P} - 0.01$
Step 2: $\underline{P} + 0.02$, $\overline{P} - 0.02$



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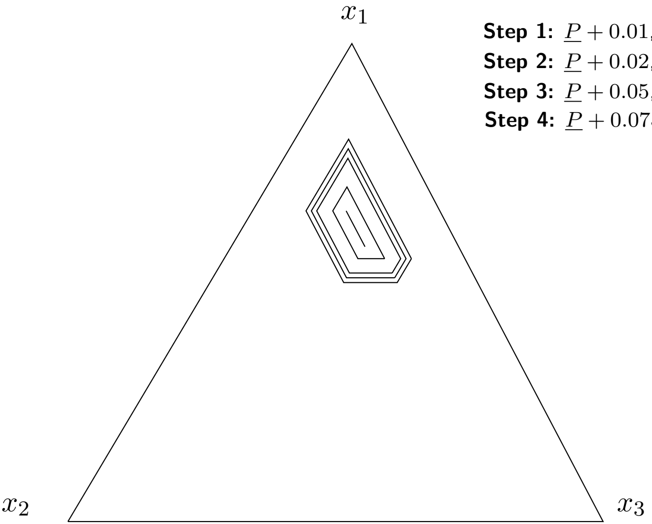
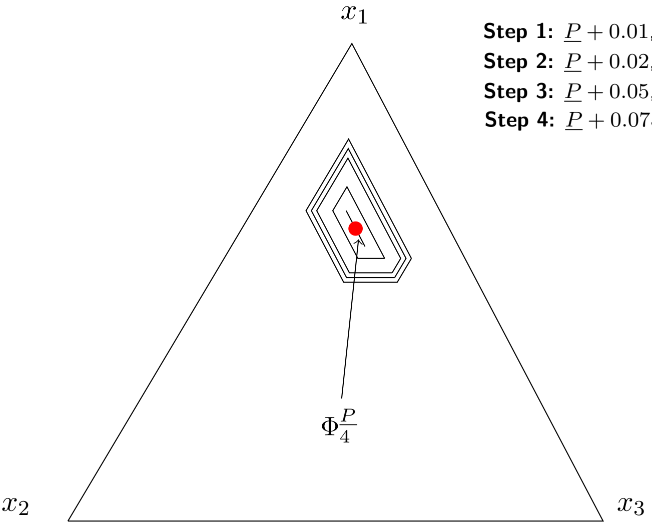


Step 1: $\underline{P} + 0.01$, $\overline{P} - 0.01$
Step 2: $\underline{P} + 0.02$, $\overline{P} - 0.02$
Step 3: $\underline{P} + 0.05$, $\overline{P} - 0.05$
Step 4: $\underline{P} + 0.075$, $\overline{P} - 0.075$

4.Contraction centroid: motivation



Step 1: $\underline{P} + 0.01$, $\overline{P} - 0.01$
Step 2: $\underline{P} + 0.02$, $\overline{P} - 0.02$
Step 3: $\underline{P} + 0.05$, $\overline{P} - 0.05$
Step 4: $\underline{P} + 0.075$, $\overline{P} - 0.075$

4.Contraction centroid

$\mathcal{M}(\underline{P})$ can be expressed as:

$$\mathcal{M}(\underline{P}) = \left\{P \mid \; P(A) = \underline{P}(A) \; \forall A \in \mathcal{L}^{=}, \; P(A) \geq \underline{P}(A) \; \forall A \in \mathcal{L}^{>} \right\}$$

where:

$$\underline{P}(A) = \overline{P}(A) \quad \text{if } A \in \mathcal{L}^{=} \qquad \underline{P}(A) < \overline{P}(A) \quad \text{if } A \in \mathcal{L}^{>}$$

We define:

$$\mathcal{M}(\underline{P})_{\alpha} = \left\{P \mid \; P(A) = \underline{P}(A) \; \forall A \in \mathcal{L}^{=}, \; P(A) \geq \underline{P}(A) + \alpha \; \forall A \in \mathcal{L}^{>} \right\}$$

$$\alpha_1 = \sup\{\alpha \mid \mathcal{M}(\underline{P})_{\alpha} \neq \emptyset\}$$

determining a coherent lower probability as

$$\underline{P}_1(A) = \min\{P(A) \mid P \in \mathcal{M}(\underline{P})_{\alpha_1}\}.$$

4.Contraction centroid

Iterating the procedure, after a finite number of steps we get:

$$\mathcal{M}(\underline{P}) \supset \mathcal{M}(\underline{P})_{\alpha_1} \supset \mathcal{M}(\underline{P}_1)_{\alpha_2} \supset \ldots \supset \{\Phi_4^{\underline{P}}\}.$$

$\Phi_4^{\underline{P}}$ is called the contraction centroid.

Properties:

- "Well defined".
- $\mathcal{M}(\underline{P})_{\alpha_1}$ coincides with the set of incenters *(whenever the interior of $\mathcal{M}(\underline{P})$ is non-empty)*.
- Simple expression when $\underline{P}$ is 2-monotone.

4.Contraction centroid: example

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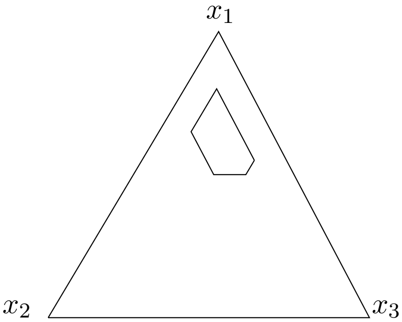
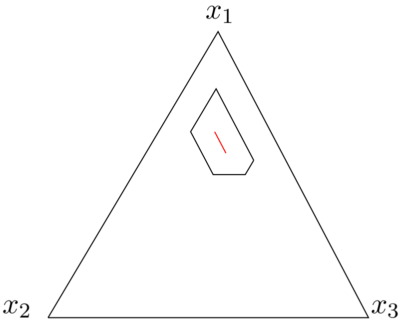
$\overline{P}(A)$	0.8	0.25	0.35	0.9	0.9	0.5
$\underline{P}(A) + \alpha$	$0.5 + \alpha$	$0.1 + \alpha$	$0.1 + \alpha$	$0.65 + \alpha$	$0.75 + \alpha$	$0.2 + \alpha$



4.Contraction centroid: example

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$
$\underline{P}(A)$	0.5	0.1	0.1	0.65	0.75	0.2
$\overline{P}(A)$	0.8	0.25	0.35	0.9	0.9	0.5
$\underline{P}(A) + \alpha$	$0.5 + \alpha$	$0.1 + \alpha$	$0.1 + \alpha$	$0.65 + \alpha$	$0.75 + \alpha$	$0.2 + \alpha$



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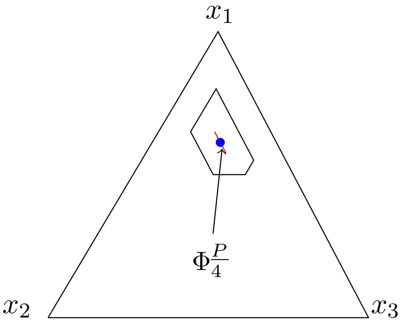
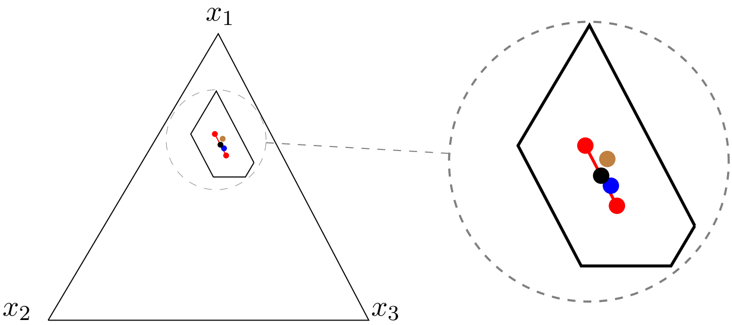


Summary



Shapley Average of extreme points
Set of incenters Contraction centroid

Open Question 1

OQ1: Centroids as game solutions?

- Interpretation of the centroids from the viewpoint of games?
- Axiomatic definition?

Inner approximations

$\underline{P}'$ is an *inner approximation* of $\underline{P}$ if:

$$\underline{P}' \geq \underline{P} \Leftrightarrow \mathcal{M}(\underline{P}') \subseteq \mathcal{M}(\underline{P}).$$

$\underline{P}'$ is a *non-dominating* inner approximation in $\mathcal{C}$ if there is not another inner approximation $\underline{P}'' \in \mathcal{C}$ satisfying $\underline{P}' \gneq \underline{P}'' \geq \underline{P}$.

LV-Incenters

We look for a $\underline{P}_{LV}$ determined by (P_0, δ) inner approximating $\underline{P}$.

LV-Incenter radius:

$$\delta_{LV} = \sup\left\{\delta \mid \exists P_0 \in \mathcal{M}(\underline{P}) \text{ satisfying } B^{\delta}_{LV}(P_0) \subseteq \mathcal{M}(\underline{P})\right\}.$$

LV-Incenter: any P_0 such that $B^{\delta_{LV}}_{LV}(P_0) \subseteq \mathcal{M}(\underline{P})$.

Properties:

- It exists iff $\underline{P}(A) < \overline{P}(A)$ for every $A \neq \emptyset, \mathcal{X}$.
- It may not be unique.
- Simple expression under 2-monotonicity.

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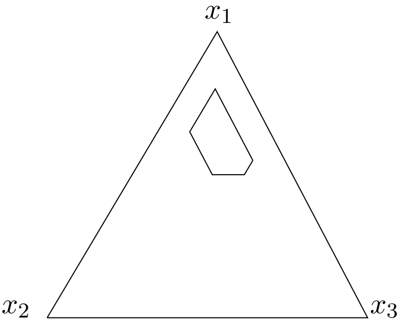

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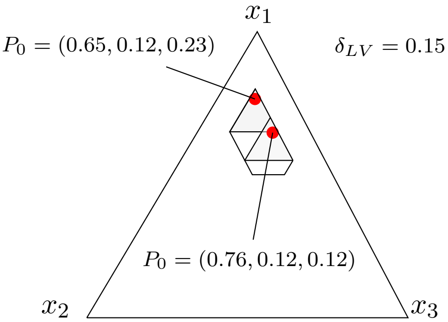

PMM-Incenters

We look for a $\underline{P}_{PMM}$ determined by (P_0, δ) inner approximating $\underline{P}$.

PMM-Incenter radius:

$$\delta_{PMM} = \sup\left\{\delta \mid \exists P_0 \in \mathcal{M}(\underline{P}) \text{ satisfying } B^{\delta}_{PMM}(P_0) \subseteq \mathcal{M}(\underline{P})\right\}.$$

PMM-Incenter: any P_0 such that $B^{\delta_{PMM}}_{PMM}(P_0) \subseteq \mathcal{M}(\underline{P})$.

Properties:

- It exists iff $\underline{P}(A) < \overline{P}(A)$ for every $A \neq \emptyset, \mathcal{X}$.
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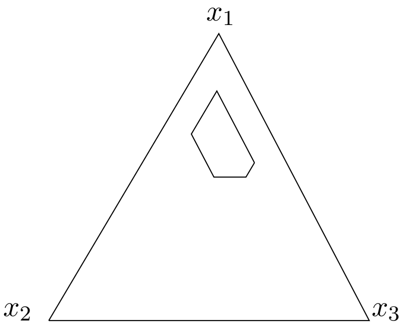

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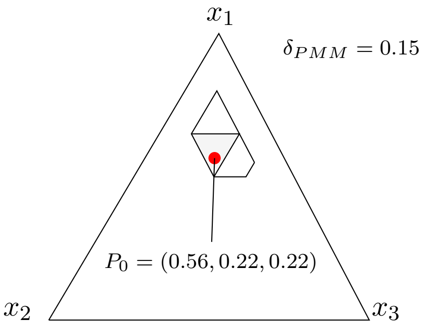


Open Question 2

OQ2: LV-PMM game solutions?

- Do they make sense from the view-point of games?
- Axiomatic definition?

Coalitional game theory VS Imprecise probabilities:
Two sides of the same coin. . . or not?

Overview

Example

Consider a game with 3 player $\mathcal{X} = \{x_1, x_2, x_3\}$. Opinions of the coalitions:

Condition 1 Player 1 wants, at least, 10% of the reward.

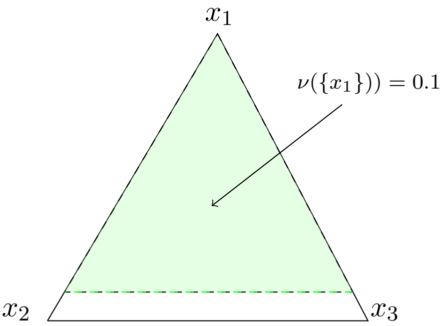
Condition 2 Player 2 wants, at least, 20% of the reward.

Condition 3 The coalition of players 1 and 2 wants, at least, 40% of the reward.

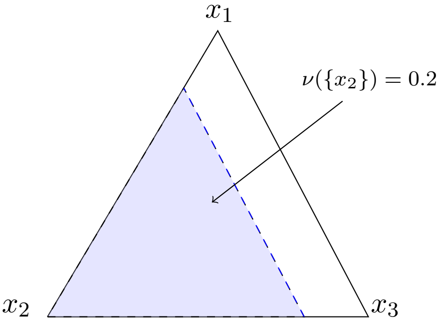
Condition 4 Player 3 wants, at least, as much as player 1.

	Game
Condition 1:	$\nu(\{x_1\}) = 0.1$
Condition 2:	$\nu(\{x_2\}) = 0.2$
Condition 3:	$\nu(\{x_1, x_2\}) = 0.4$
Condition 4:	??

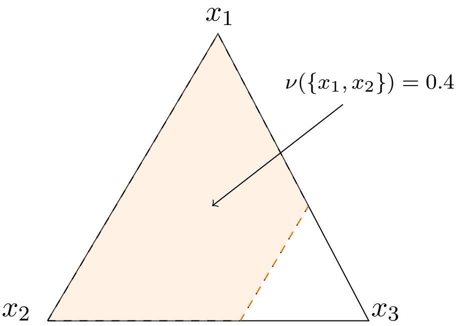
Example



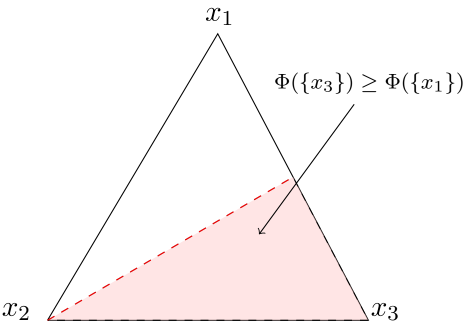
Example



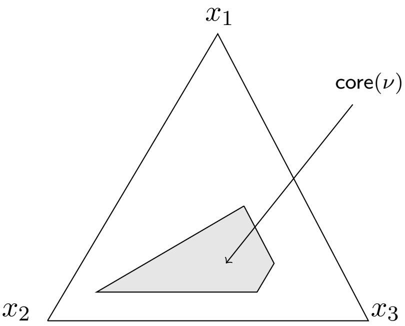
Example



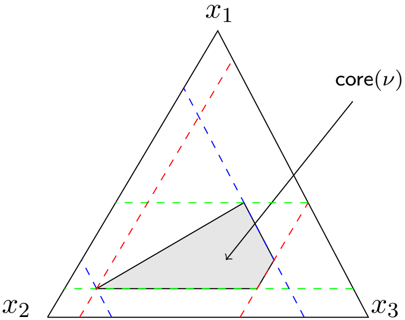
Example



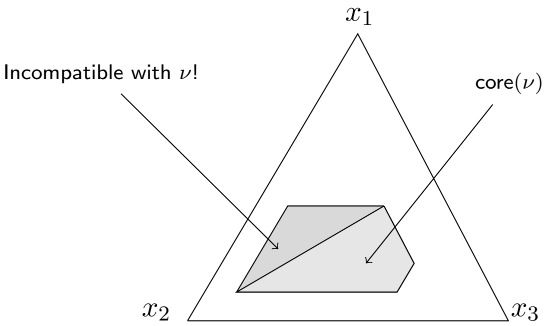
Example



Example



Example



Lower previsions

- **Gamble:** bounded function $f : \mathcal{X} \to \mathbb{R}$. Set of gambles: $\mathcal{L}(\mathcal{X})$.
- **Lower prevision:** a function $\underline{Q} : \mathcal{L}(\mathcal{X}) \to \mathbb{R}$.
- **Conjugate upper prevision:** $\overline{Q}(f) = -\underline{Q}(-f)$.
- **Credal set:**

$$\mathcal{M}(\underline{Q}) = \{P \mid P(f) \geq \underline{Q}(f)\ \forall f \in \mathcal{L}(\mathcal{X})\}.$$

- $\underline{Q}$ **avoids sure loss** if $\mathcal{M}(\underline{Q}) \neq \emptyset$.
- **Coherence:** $\underline{Q}(f) = \min\{P(f) \mid P \in \mathcal{M}(\underline{Q})\}$.

Coherent lower previsions VS credal sets

$$\begin{tikzcd}
\boxed{\text{Coherent lower previsions}} \arrow[rr, leftrightarrow] & & \boxed{\text{Credal Sets}}
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\underline{Q}(f) = \min\{P(f) \mid P \in \mathcal{M}(\underline{Q})\}} \arrow[r, bend left] \arrow[d] & \boxed{\mathcal{M}(\underline{Q}) = \{P \mid P(f) \geq \underline{Q}(f)\}} \arrow[l, bend left] \\
\boxed{\underline{P}(A) = \underline{Q}(I_A) = \min\{P(A) \mid P \in \mathcal{M}(\underline{Q})\}} &
\end{tikzcd}$$

Important: Different coherent lower previsions may induce the same coherent lower probability

Example

Consider a game with 3 player $\mathcal{X} = \{x_1, x_2, x_3\}$. Opinions of the coalitions:

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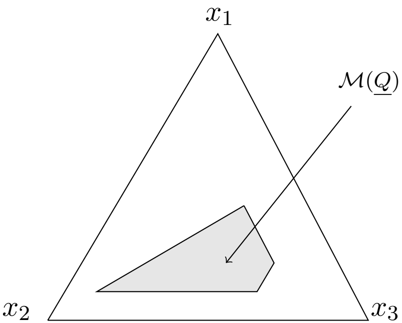
Condition 3 The coalition of players 1 and 2 wants, at least, 40% of the reward.

Condition 4 Player 3 wants, at least, as much as player 1.

	Game		**Lower prevision**
Condition 1:	$\nu(\{x_1\}) = 0.1$	$\longrightarrow$	$\underline{Q}\big(I_{\{x_1\}}\big) = 0.1$
Condition 2:	$\nu(\{x_2\}) = 0.2$	$\longrightarrow$	$\underline{Q}\big(I_{\{x_2\}}\big) = 0.2$
Condition 3:	$\nu(\{x_1, x_2\}) = 0.4$	$\longrightarrow$	$\underline{Q}\big(I_{\{x_1,x_2\}}\big) = 0.4$
Condition 4:	??	$\longrightarrow$	$\underline{Q}\big(I_{\{x_3\}} - I_{\{x_1\}}\big) = 0$

Example



1.Shapley value

Credal set	Coherent lower prevision	Coherent lower probability	Shapley value

$$
\begin{tikzcd}
\mathcal{M}(\underline{Q}) \arrow[r, bend left] & \underline{Q}(f) \arrow[l, bend left] \arrow[r] & \underline{P}(A) := \underline{Q}(I_A) \arrow[r] & \Phi_1^{\underline{Q}} := \Phi_1^{\underline{P}}
\end{tikzcd}
$$

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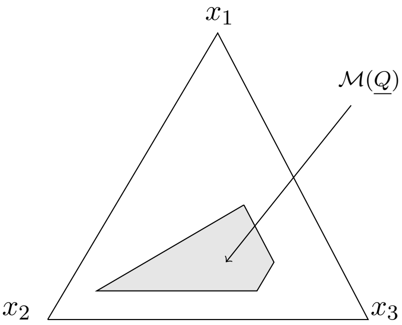
Credal set	Coherent lower prevision	Coherent lower probability	Shapley value

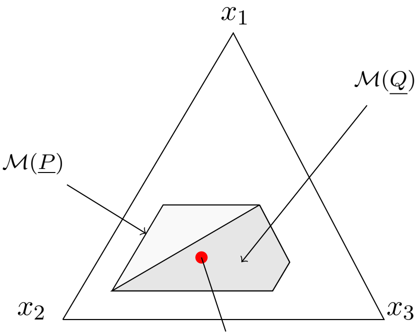
$$\begin{tikzcd} \mathcal{M}(\underline{Q}) \arrow[r, bend left] & \underline{Q}(f) \arrow[l, bend left] \arrow[r] & \underline{P}(A) := \underline{Q}(I_A) \arrow[r] & \Phi_1^{\underline{Q}} := \Phi_1^{\underline{P}} \end{tikzcd}$$



1.Shapley value

Credal set	Coherent lower prevision	Coherent lower probability	Shapley value

$$\mathcal{M}(\underline{Q}) \rightleftarrows \underline{Q}(f) \longrightarrow \underline{P}(A) := \underline{Q}(I_A) \longrightarrow \Phi_1^{\underline{Q}} := \Phi_1^{\underline{P}}$$



$$\Phi_1^{\underline{Q}} = (0.21\overline{6}, 0.46\overline{6}, 0.31\overline{6})$$

Open Question 3

OQ3: Shapley with lower previsions

- Does it make sense?
- Alternative expression?
- Rewriting the axiomatic properties?

2.Average of the extreme points

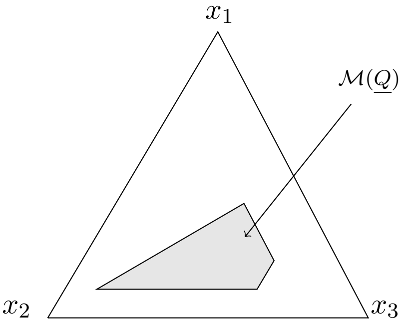
If $\mathcal{M}(\underline{Q})$ is a polytope, it has a finite number of extreme points $P_1, \ldots, P_k$:

$$\Phi_2^{\underline{Q}}(\{x\}) = \frac{1}{k}\sum_{i=1}^{k} P_i(\{x\}).$$

2.Average of the extreme points

If $\mathcal{M}(\underline{Q})$ is a polytope, it has a finite number of extreme points $P_1, \ldots, P_k$:

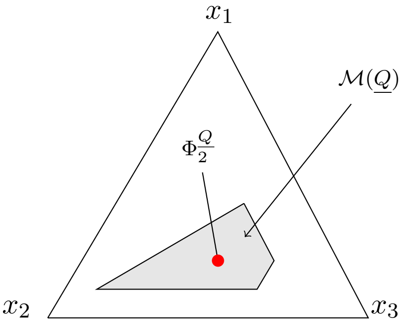
$$\Phi_2^{\underline{Q}}(\{x\}) = \frac{1}{k}\sum_{i=1}^{k} P_i(\{x\}).$$



2.Average of the extreme points

If $\mathcal{M}(\underline{Q})$ is a polytope, it has a finite number of extreme points $P_1, \ldots, P_k$:

$$\Phi_{\frac{Q}{2}}(\{x\}) = \frac{1}{k}\sum_{i=1}^{k} P_i(\{x\}).$$

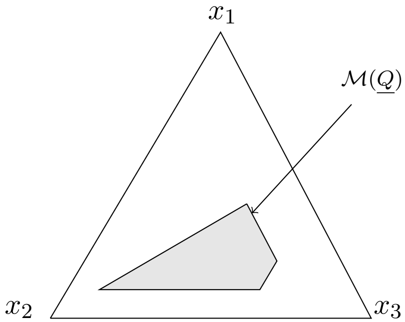


3.Incenter

Incenter radius:

$$\alpha_I = \sup\left\{\alpha \mid \exists P_0 \in \mathcal{M}(\underline{Q}) \text{ satisfying } B_o^{\alpha}(P_0) \subseteq \mathcal{M}(\underline{Q})\right\}.$$

Incenter: any $\Phi_3^{\underline{Q}}$ satisfying $B_o^{\alpha_I}\left(\Phi_3^{\underline{Q}}\right) \subseteq \mathcal{M}(\underline{Q})$.



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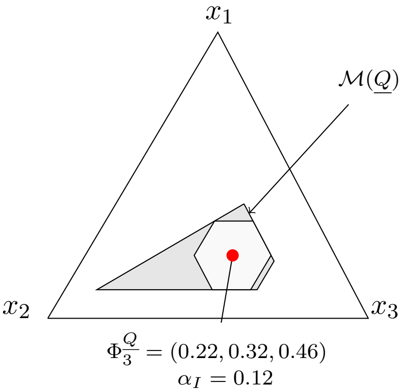
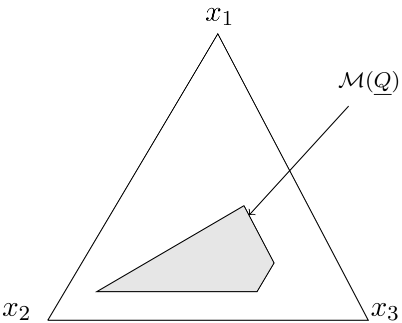


4.Contraction centroid



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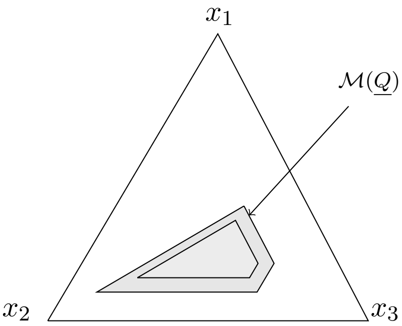

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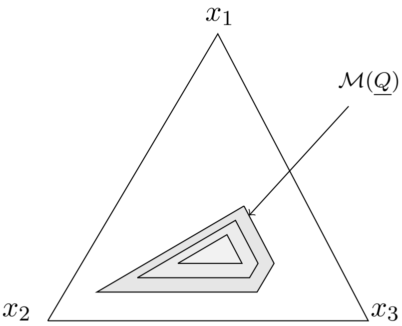

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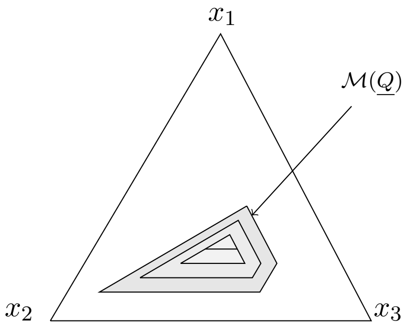
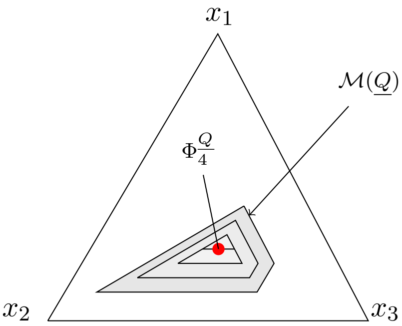
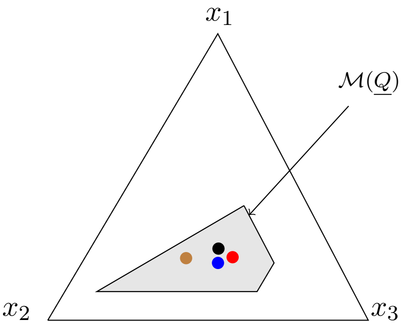


4.Contraction centroid



Summary



Shapley

Set of incenters

Average of extreme points

Contraction centroid

Open Question 4

OQ4: general credal sets

- What if $\mathcal{M}(\underline{Q})$ is not a polytope?
- Average of extreme points?
- Contraction centroid?

Overview

Lower probabilities

Coalitional games

Game solutions - Centroids

Lower previsions VS coalitional games

Conclusions

At a glance

Lower probabilities VS coalitional games: equivalent ...

Coalitional game theory	**Lower probabilities**
Normalised game (ν)	Lower probability ($\underline{P}$)
Conjugate ($\bar{\nu}$)	Conjugate upper probability ($\overline{P}$)
Balanced game	Lower probability avoiding sure loss
Exact game	Coherent lower probability
Core (core(ν))	Credal set ($\mathcal{M}(\underline{P})$)
Convex game	2-monotone lower probability
Game solution	Centroid of the credal set

...but lower previsions are more informative!

Open Question 5

OQ5: IP models

Coalitional games and . . .

- . . . set of desirable gambles?
- . . . choice functions?
- . . .

References (I)

Shapley and Banzhaf values as probability transformations. E. Miranda, I. Montes, IJUFKS 2018.

International Journal of Uncertainty,
Fuzziness and Knowledge-Based Systems
Vol. 26, No. 6 (2018) 917–947

DOI: 10.1142/S0218488518500411

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Shapley and Banzhaf Values as Probability Transformations

Enrique Miranda* and Ignacio Montes†

Centroids of the core of exact capacities: a comparative study. E. Miranda, I. Montes, Submitted.

Centroids of the core of exact capacities: a comparative study

Enrique Miranda[1*] and Ignacio Montes[1†]

[1*]Department of Statistics and O.R., University of Oviedo, Federico García Lorca, Oviedo, 33007, Asturias, Spain.

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- *Unifying neighbourhood and distortion models: Parts I and II*. I. Montes, E. Miranda, S. Destercke, JIGS 2020.

INTERNATIONAL JOURNAL OF GENERAL SYSTEMS
https://doi.org/10.1080/03081079.2020.1778682

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Unifying neighbourhood and distortion models: part I – new results on old models

Unifying neighbourhood and distortion models: part II – new models and synthesis

Ignacio Montes[a], Enrique Miranda[a] and Sébastien Destercke[b]

- *Inner approximations of coherent lower probabilities. . . .* E. Miranda, I. Montes, A. Presa, In preparation.

Inner approximations of coherent lower probabilities and their application to decision making problems

Enrique Miranda[1*†], Ignacio Montes[1†] and Andrés Presa[1†]

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