Function-Coherent Gambles

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Exaloan

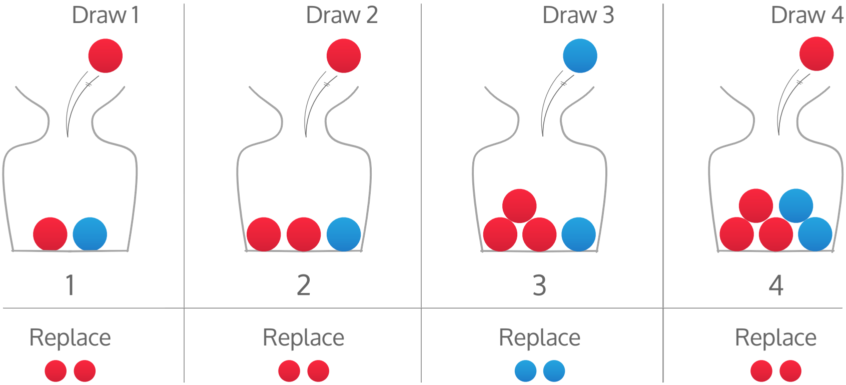
Plan

1
Ergodicity Breaking

2
Varieties of Discounting

3
Function Coherence

Pólya's Urn



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Draw 2
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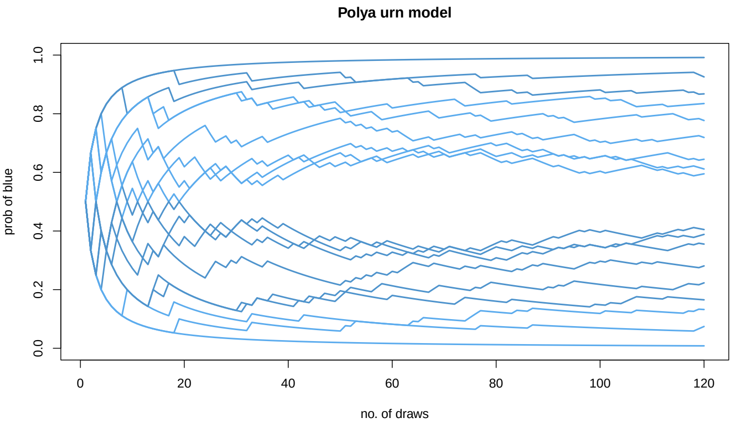
Pólya's Urn



- Proportion of red balls converges to *some* value between 0 and 1
- This limiting value is random
- Any value between 0 and 1 is **equally likely** as a limiting proportion

Pólya's Urn

$$\mathbb{E}[f] = \frac{1}{2} \neq \lim_{t \to \infty} \mu[f, t]\text{, for } \textit{any} \text{ sequence}$$

The ensemble average $(\frac{1}{2})$ *is not representative of any individual trajectory.*

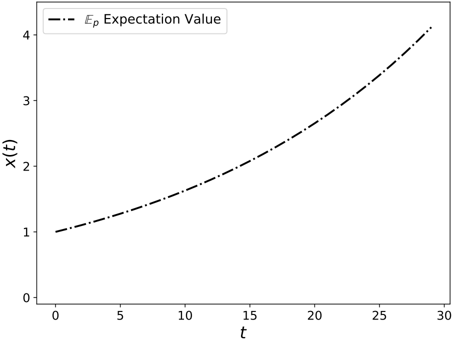
No individual urn represents the "typical" behavior

Discounting Repeated Gambles

A Simple Gamble f

Heads: ***increase*** *your stake x by 50%*
Tails: ***decrease*** *your stake x by 40%*

Discounting Repeated Gambles



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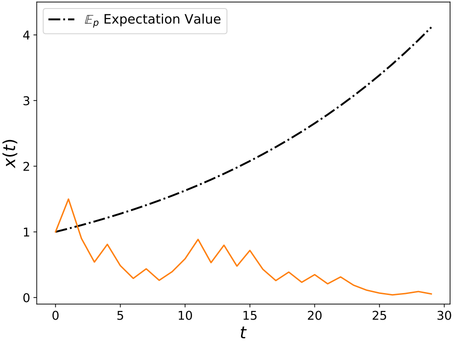
If $p = \frac{1}{2}$ and **x = 1** at t_0, then $\mathbb{E}_p[f] =$ **1.05**

Should you stake €1 on 30 tosses?

Discounting Repeated Gambles



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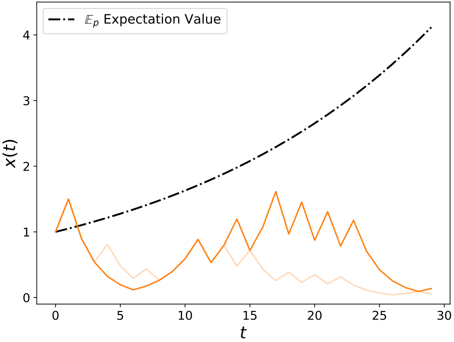

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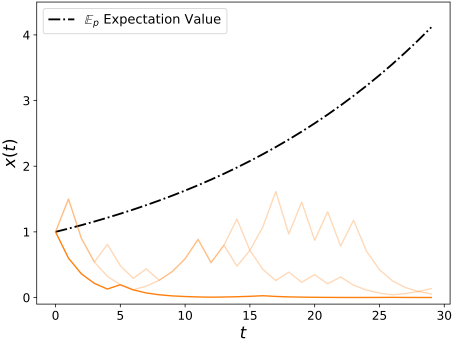

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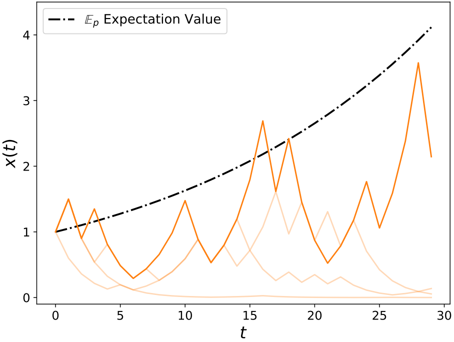

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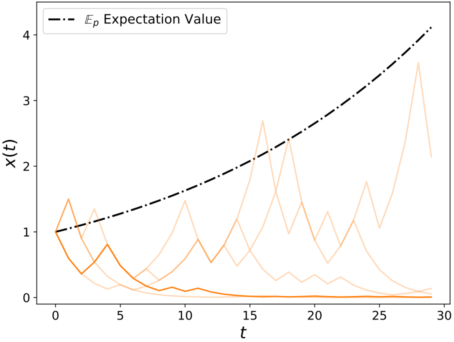

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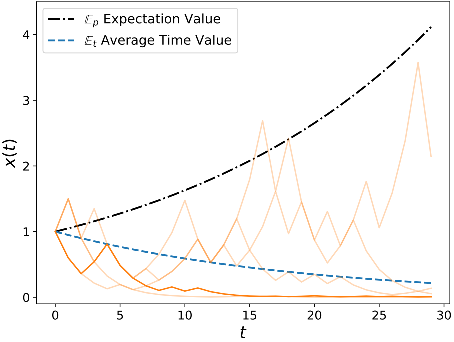

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Should you stake €1 on 30 tosses?

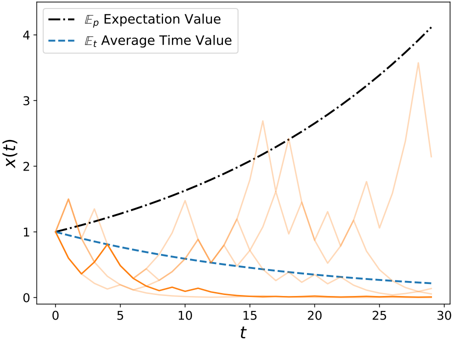
$\mathbb{E}_p[f] \approx$ *€4.12, after 30 tosses*

expectation value

$\mathbb{E}_t[f] \approx$ *€0.22, after 30 tosses*

most probable value

Discounting Repeated Gambles

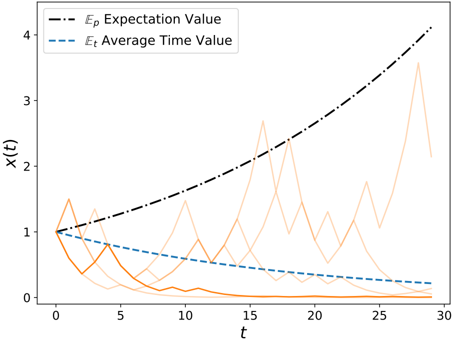


UPSHOT

Expectation values ***do not always reflect*** *what happens over time:*

time average growth $\not\equiv$ expected rate of change

Discounting Repeated Gambles



BIRKHOF'S EQUALITY

$$\lim_{t \to \infty} \mu\left[f(\omega, t)\right] = \mathbb{E}\left[f(\omega)\right]$$

Describes the conditions under which the ***expectation value*** *of a repeated gamble is equivalent to long-term time average of a single sequence of gambles.*

The **dogma of ergodicity** does not question whether this equality holds.

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Rationality Wars

It seems impossible to reach any definitive conclusions concerning human rationality in the absence of a detailed analysis of the sensitivity of the criterion and the cost involved in evaluating the alternatives. When the difficulty (or the costs) of the evaluations and the consistency (or the error) of the judgments are taken into account, a [transitivity-violating method] may prove superior.

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(Tversky 1969)

Preference Reversals in the Wild

Canadian Jay

Option A: 1 raisin (28cm distance)

Option B: 2 raisins (42cm distance)

Option C: 3 raisins (56cm distance)

$$A \succ B \succ C$$

yet

$$C \succ A$$

(*Waite* 2001)

Preference Reversals in the Wild

Honeybees

Option A

Option B

Option C

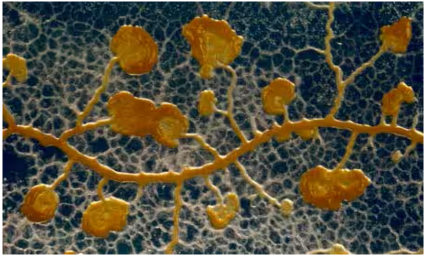
$$A \succ B \succ C$$

yet

$$C \succ A$$

(*Shafir* 1994)

Preference Reversals in the Wild



Slime Mold

Option A

Option B

Option C

$$A \succ B \succ C$$

yet

$$C \succ A$$

(*Latty & Beekman* 2011)

Preference Reversals in the Wild

Humans

Exponential Discounting

$$present\_value \times e^{-rt}$$

- Discount rate constant over time
- Never produces preference reversals

Hyperbolic Discounting

$$present\_value/(1+kt)$$

- Discount rate decreases as time-delay increases
- Regularly produces preference reversals

Preference Reversals in the Wild

Humans

Hyperbolic Discounting

Scenario 1: Choose

✓ **A)** $100 today
B) $110 tomorrow

Scenario 2: Choose

A) $100 in 30 days
✓ **B)** $110 in 31 days

Preference Reversals in the Wild

Humans

Hyperbolic Discounting

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Scenario 2: Choose

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Average Americans live ~4 years longer with hyperbolic discounting
compared to exponential discounting

(*Strulik & Schünemann* 2018)

Varieties of Discounting

Hyperbolic	$D_H(t) = \frac{1}{1+kt}, \quad k > 0$
Quasi-Hyperbolic	$D_Q(t) = \{1 \text{ if } t = 0, \quad \beta\delta^t \text{ if } t > 0\}$
Generalized Hyperbolic	$D_G(t) = \frac{1}{(1+kt)^p}, \quad k > 0, p > 0$
Scale-Dependent	$D_S(t, x) = D(t)^{\eta(x)}$
State-Dependent	$D_W(t, x) = e^{-r(s)t}$
Hybrid	$D_{hyb}(t) = \lambda D_1(t) + (1-\lambda)D_2(t), \quad \lambda \in [0, 1]$

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Varieties of Discounting
Preference Change

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Desirable Gambles

$\mathbb{D}$ is a ***coherent set of desirable gambles*** (Williams 1975; Walley 2000) iff:

A1. If $f < 0$, then $f \notin \mathbb{D}$ **(Avoid partial losses)**

A2. If $f \geqslant 0$, then $f \in \mathbb{D}$ **(Accept partial gains)**

A3. If $f \in \mathbb{D}$, then $\lambda f \in \mathbb{D}$ $(\lambda \geqslant 0)$ **(Positive scale invariance)**

A4. If $f \in \mathbb{D}$ and $g \in \mathbb{D}$, then $f + g \in \mathbb{D}$ **(Combination)**

f, g are bounded gambles, assessed pointwise; $\lambda \geqslant 0$

Thm: If $\mathbb{D}$ is coherent, then there is a $\underline{\mathbb{E}}(f)$ $(\forall f \in \mathbb{D})$

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Desirable Gambles

Idea

Desire satisfaction is concave.

But 'desirability' is linear:

- scale invariance
- additive

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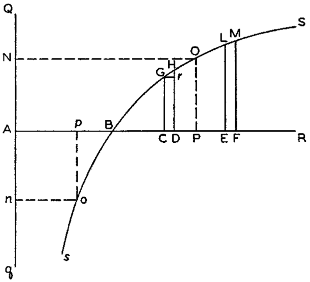

Bernoulli (1738) utility of wealth

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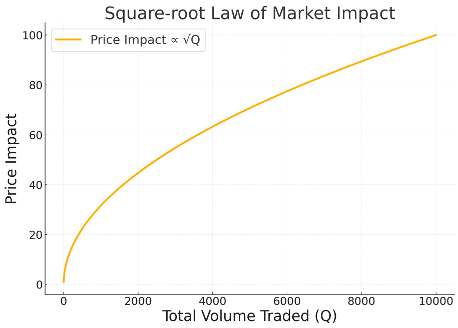
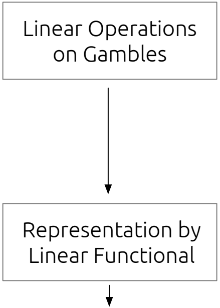


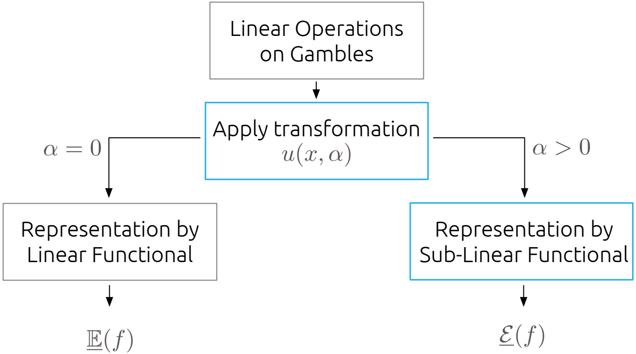
Classical Coherence



$\underline{\mathbb{E}}(f)$

✓ *No sure loss*
✓ *Monotonicity*
✓ *Cardinal utility*

U Coherence



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F3. For $f, g \in \mathbb{D}$ and nonnegative λ, μ where

$$h = u^{-1}\left(\lambda u(f) + \mu u(g)\right)$$

is **defined**, $h \in \mathbb{D}$. (u-**Convexity**)

Function-Coherent Gambles

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$\mathbb{D} \subseteq X$ is ***function-coherent*** iff:

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F3a. $u : X \mapsto V$ is a strictly increasing and continuous bijection onto its image, $u(X) \subseteq V$.

F3b. The image $u(X)$ is convex. Specifically, for any $f, g \in \mathbb{D}$ and any non-negative scalars λ, μ, the linear combination $\lambda u(f) + \mu u(g)$ is always in the domain of u^{-1}

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PROPERTIES

- **Non-triviality**:
  - $\mathbb{D}$ is non-empty
  - Every f with $u(f) \geqslant 0$ is acceptable
- **Upward Closure**: If $f \in \mathbb{D}$ and $g \in X$ satisfies $g(s) \geqslant f(s)$ for all states s, then $g \in \mathbb{D}$.
- **Transform Convexity**: The u-transformed set

$$U(\mathbb{D}) := \{u(f) : f \in \mathbb{D}\}$$

is a convex cone.

- **Transform Invariance**: For strictly increasing ϕ with $\phi(0) = 0$, if $\tilde{u} = \phi \circ u$ then

$$\{f \in X : \tilde{u}(f) \geqslant 0\} = \{f \in X : u(f) \geqslant 0\} = \mathbb{D}.$$

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  - $\mathbb{D}$ possesses a non-empty interior

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Thm 1 (representation)

There exists a continuous linear functional $\ell : V \to \mathbb{R}$, unique up to positive scaling, such that

$$f \in \mathbb{D} \iff \ell(u(f)) \geqslant 0.$$

Thm 2 (closure under limits)

Suppose X is a topological vector space and the utility function $u : X \to \mathbb{R}$ is continuous.

An acceptance set is $\mathbb{D} = \{f \in X : u(f) \geqslant 0\}$. If $\{f_n\}$ is a sequence in $\mathbb{D}$ that converges to some $f \in X$, then $f \in \mathbb{D}$.

Representation and Risk Measures

$$f \in \mathbb{D} \;\Leftrightarrow\; \ell(u(f)) \geqslant 0$$

Three Implications

- ρ is a **generalized risk measure** incorporating non-linear utility
- **Preference-belief Decomposition**: The composition $\ell \circ u$ cleanly separates:

  Preferences encoded by utility u, and
  Beliefs captured by aggregator ℓ.

- **Order invariance**: relative preferences, not absolute risk values, determine choice behavior.

Function-coherent gambles

Gregory Wheeler[1]

[1]Frankfurt School of Finance & Management, Germany

Abstract

The desirable gambles framework provides a foundational approach to imprecise probability theory but relies heavily on linear utility assumptions. This paper introduces *function-coherent gambles*, a gen-

classification-theoretic formulations of non-linear desirability, the approach in [25] and here maintains an axiomatic foundations and connects non-linear utility to representation via continuous functionals. In [25], two routes were initially explored. The first preserves the additive structure of the desirable gambles framework

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A4. If $f \in \mathbb{D}$ and $g \in \mathbb{D}$, then $f + g \in \mathbb{D}$ (**combination**)

f, g are bounded gambles, assessed pointwise; $\lambda \geqslant 0$

Function-coherence

$\mathbb{D}$ is ***function-coherent*** iff:

F1. If $f < 0$, then $f \notin \mathbb{D}$ (**Avoid partial losses**)
F2. If $f \geqslant g$ and $g \in \mathbb{D}$, then $f \in \mathbb{D}$ (**Monotonicity**)
F3. For $f, g \in \mathbb{D}$ and nonnegative λ, μ where

$$h = u^{-1}\left(\lambda u(f) + \mu u(g)\right)$$

is **defined**, $h \in \mathbb{D}$. (u-**Convexity**)

Function-Coherent Gambles

Coherence Axioms

$\mathbb{D}$ is a ***coherent set of desirable gambles*** iff:

- A1. If $f < 0$, then $f \notin \mathbb{D}$ (**Avoid partial losses**)
- A2. If $f \geqslant 0$, then $f \in \mathbb{D}$ (**Accept partial gains**)
- A3. If $f \in \mathbb{D}$, then $\lambda f \in \mathbb{D}$ (**Pos. scale invariance**)
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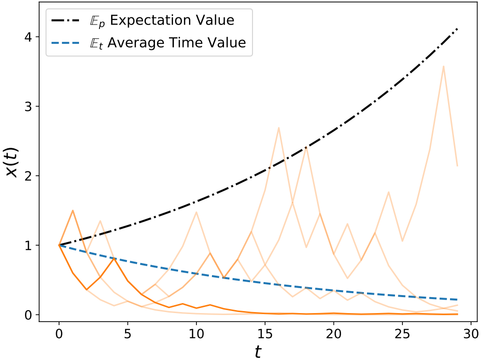
Function-Coherent Gamble Dynamics



Updates wealth by a factor of $1 + f(\omega)$ in state ω.
After n independent repetitions, wealth evolves as

$$w' = w \prod_{i=1}^{n} (1 + f(\omega_i)).$$

Function-coherence

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Function-Coherent Gamble Dynamics

Non-additive Sequence Dynamics

Updates wealth by a factor of $1 + f(\omega)$ in state ω.

After n independent repetitions, wealth evolves as

$$w' = w \prod_{i=1}^{n} \big(1 + f(\omega_i)\big).$$

The long-run performance is determined by the geometric mean, or equivalently, by the time-average of logarithmic returns:

$$\frac{1}{n}\sum_{i=1}^{n} \log\big(1 + f(\omega_i)\big).$$

Idea: *Introduce a non-linear combination operator.*

Function-coherence

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F1. If $f < 0$, then $f \notin \mathbb{D}$ (**Avoid partial losses**)

F2. If $f \geqslant g$ and $g \in \mathbb{D}$, then $f \in \mathbb{D}$ (**Monotonicity**)

F3. For $f, g \in \mathbb{D}$ and nonnegative λ, μ where

$$h = \upsilon^{-1}\left(\lambda \upsilon(f) + \mu \upsilon(g)\right)$$

is defined, $h \in \mathbb{D}$. (υ-**Convexity**)

Function-Coherent Gamble Dynamics

Non-linear combination

(F4) **Nonlinear Combination:** If $f \in \mathbb{D}$ and $g \in \mathbb{D}$, then their nonlinear combination

$$f(\omega) \oplus g(\omega) := \phi^{-1}\Big(\phi\big(f(\omega)\big) + \phi\big(g(\omega)\big)\Big)$$
$$= (1 + f(\omega))\,(1 + g(\omega)) - 1,$$

is also in $\mathbb{D}$.

Log-return transformation

Define

$$L(f) := \log(1 + f)$$

Then,

$$L(f \oplus g) = L(f) + L(g)$$

Function-Coherent Gamble Dynamics

Non-linear combination

(F4) **Nonlinear Combination:** If $f \in \mathbb{D}$ and $g \in \mathbb{D}$, then their nonlinear combination

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$$= (1 + f(\omega))\,(1 + g(\omega)) - 1,$$

is also in $\mathbb{D}$.

Key idea: The $\oplus$ operator converts multiplicative effects into an additive structure in the log-domain.

Log-return transformation

Define

$$L(f) := \log(1 + f)$$

Then,

$$L(f \oplus g) = L(f) + L(g)$$

Function-Coherent Gambles

Function-coherence

$\mathbb{D} \subseteq X$ is ***function-coherent*** iff:

F1. If $f < 0$, then $f \notin \mathbb{D}$ (**Avoid partial losses**)

F2. If $f \geqslant g$ and $g \in \mathbb{D}$, then $f \in \mathbb{D}$ (**Monotonicity**)

F3. For $f, g \in \mathbb{D}$ and nonnegative λ, μ where

$$h = u^{-1}\left(\lambda u(f) + \mu u(g)\right)$$

is **defined**, $h \in \mathbb{D}$. (u-**Convexity**)

F4. If $f \in \mathbb{D}$ and $g \in \mathbb{D}$, then their nonlinear combination

$$f(\omega) \oplus g(\omega) := \phi^{-1}\Big(\phi\big(f(\omega)\big) + \phi\big(g(\omega)\big)\Big)$$

is also in $\mathbb{D}$. (**Nonlinear combination**)

Function-Coherent Gambles

Function-coherence

$\mathbb{D} \subseteq \mathcal{X}$ is ***function-coherent*** iff:

F1. If $f < 0$, then $f \notin \mathbb{D}$ (**Avoid partial losses**)

F2. If $f \geqslant g$ and $g \in \mathbb{D}$, then $f \in \mathbb{D}$ (**Monotonicity**)

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F4. If $f \in \mathbb{D}$ and $g \in \mathbb{D}$, then their nonlinear combination

$$f(\omega) \oplus g(\omega) := \phi^{-1}\Big(\phi\big(f(\omega)\big) + \phi\big(g(\omega)\big)\Big)$$

is also in $\mathbb{D}$. (**Nonlinear combination**)

Thm 3 (log-domain additivity)

Let f and g be gambles satisfying $f(\omega), g(\omega) > -1$ for all ω. Then, for every state $\omega \in \Omega$,

$$\log\Big(1 + (f \oplus g)(\omega)\Big) = \log\big(1 + f(\omega)\big) + \log\big(1 + g(\omega)\big).$$

Thm 4 (Function-coherence preserv'tn)

Let $\mathcal{X}$ be a space of gambles on Ω with $f(\omega) > -1$ for all $f \in \mathcal{X}$ and $\omega \in \Omega$. Suppose the acceptance set $\mathbb{D} \subseteq \mathcal{X}$ satisfies (F1 to F3), the regularity conditions, and (F4).

Then there exists a continuous linear functional η on a suitable vector space V (of log-returns) such that for every gamble $f \in \mathcal{X}$,

$$f \in \mathbb{D} \iff \ell\big(L(f)\big) \geqslant 0.$$

Some well-behaved utility functions

Several important classes of well-behaved utility functions emerge:

1. **Power Utilities**: For $\gamma \neq 0$,

$$u_\gamma(x) = \begin{cases} \frac{x^\gamma}{\gamma}, & x \geqslant 0 \\ -\infty, & x < 0 \end{cases}$$

   Leading to the combination operator:

$$f \oplus_{u_\gamma} g = (f^\gamma + g^\gamma)^{1/\gamma}$$

2. **Exponential Utilities**: For $\alpha > 0$, as studied by (Arrow 1965):

$$u_\alpha(x) = 1 - e^{-\alpha x}$$

   With combination operator:

$$f \oplus_{u_\alpha} g = -\frac{1}{\alpha} \log \left(e^{-\alpha f} + e^{-\alpha g} - 1 \right)$$

Some well-behaved utility functions

3. *Logarithmic Utility*: Our previous case from (Wheeler 2021):

$$u(x) = \log(1 + x)$$

With combination operator:

$$f \oplus g = (1 + f)(1 + g) - 1$$

Risk Measurement and Dynamic Properties

Definition (Induced Risk Measure)

For a utility function u with well-behaved combination operator $\oplus_u$, the induced risk measure is:

$$\rho_u(f) := -\ell(u(f))$$

where ℓ is the linear functional.

Risk Measurement and Dynamic Properties

1. **Power Utility Risk Measures** ($\gamma \in (0, 1)$):

$$\rho_\gamma(f) = -\mathbb{E}\left[\frac{f^\gamma}{\gamma}\right]$$

exhibits decreasing relative risk aversion. Under power utility, an agent's risk aversion decreases as wealth increases (Merton 1971; Acerbi 2002).

Risk Measurement and Dynamic Properties

2. **Exponential Risk Measures**:

$$\rho_\alpha(f) = \frac{1}{\alpha} \log \mathbb{E}[e^{-\alpha f}]$$

exhibits constant absolute risk aversion, recovering the entropic risk measure. Under exponential risk utility, an agent's risk aversion is absolute regardless of wealth (Föllmer and Schied 2002).

Risk Measurement and Dynamic Properties

3. **Logarithmic Risk Measures**:

$$\rho_{\log}(f) = -\mathbb{E}[\log(1 + f)]$$

exhibits constant relative risk aversion and naturally captures multiplicative risks.

Like power utility, logarithmic risk aversion is proportional. Unlike power utility, logarithmic risk captures proportional multiplicative risk (i.e., compounding), which is a property of gambles rather than a psychological appetite for risk (Kelly 1956; Peters 2019; Wheeler 2018).

Function-coherent gambles with non-additive sequential dynamics

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Abstract

The desirable gambles framework provides a rigorous foundation for imprecise probability theory but relies heavily on linear utility via its coherence axioms. In our related work, we introduced function-coherent gambles to accommodate non-linear utility. However, when repeated gambles are played over time—especially in intertemporal choice where rewards compound multiplicatively—the standard additive combination axiom fails to capture the appropriate long-run evaluation. In this paper we extend the framework by relaxing the ad- essential rationality conditions [27]. A recent generalization replaces the standard convexity closure with a more abstract closure operator to model non-linearity directly in the acceptance set [15]. In contrast, the present paper develops a concrete and operationally motivated specialization of this approach. We introduce a novel combination operator that preserves coherence while accommodating non-linear utility, addressing a fundamental limitation in the standard desirable gambles framework. This operator, defined as $f \oplus g = (1+f)(1+g) - 1$, naturally captures the multiplicative dynamics of compound growth while maintaining essential rationality

Plan

1
Ergodicity Breaking
Dynamics

2
Varieties of Discounting
Preference Change

3
Function Coherence
Belief / Uility
&
Dynamics

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