

# Imprecision

(not as a problem, but as part of the solution)

Gert de Cooman

Foundations Lab  
for imprecise probabilities



I have learnt from (talking to and working with) many ...



Teddy Seidenfeld



Peter Walley



Glenn Shafer



Volodya Vovk



Philip Dawid



Marco Zaffalon



Enrique Miranda



Matthias Troffaes



Erik Quaeghebeur



Jasper De Bock

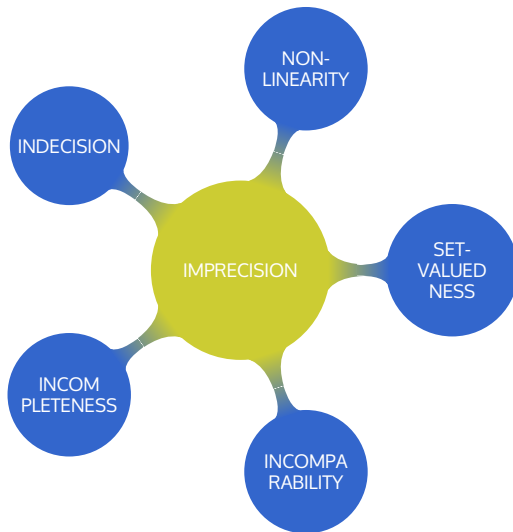


Arthur Van Camp

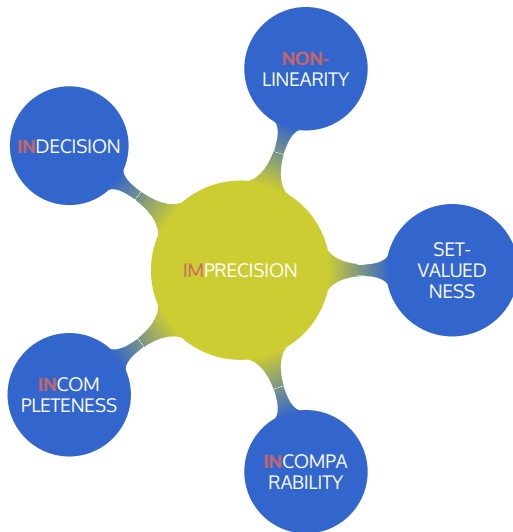


Floris Persiau

# Imprecision comes in many guises ...



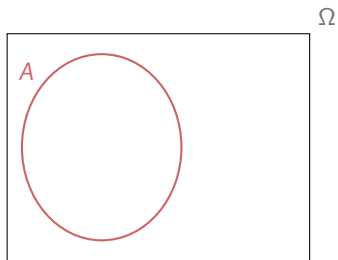
# Imprecision comes in many guises ...



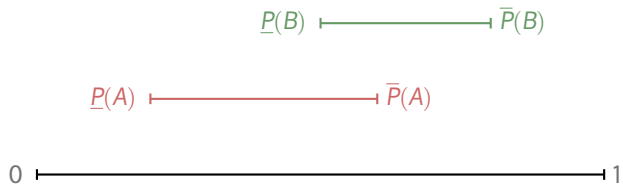
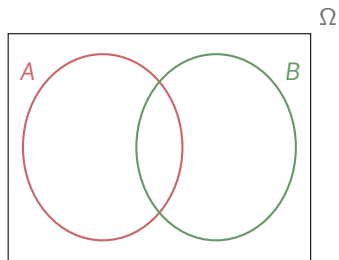


# PROBABILITY INTERVALS

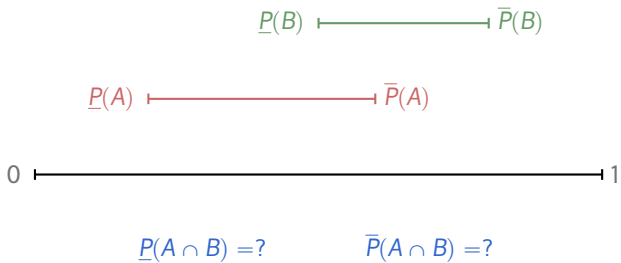
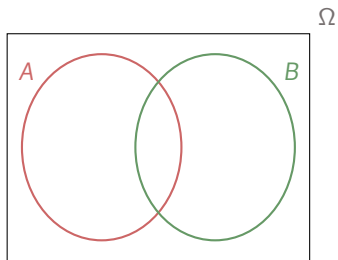
# Probability intervals



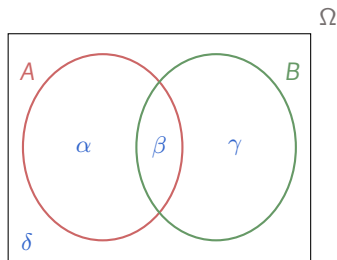
# Probability intervals



# Probability intervals



# Probability intervals



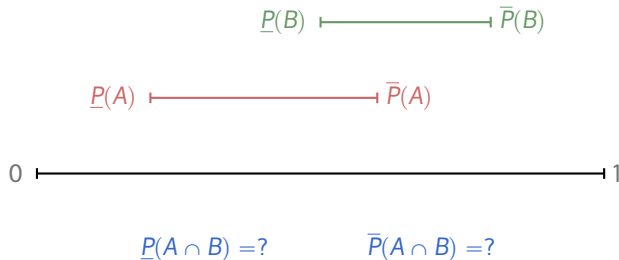
Maximise and minimise  $\beta$  under the constraints:

$$\alpha + \beta + \gamma + \delta = 1$$

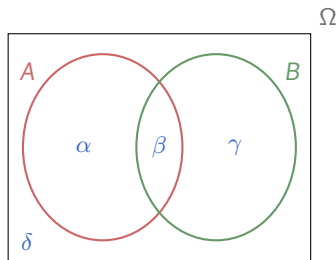
$$\alpha, \beta, \gamma, \delta \geq 0$$

$$\underline{P}(A) \leq \alpha + \beta \leq \bar{P}(A)$$

$$\underline{P}(B) \leq \beta + \gamma \leq \bar{P}(B)$$



# Probability intervals



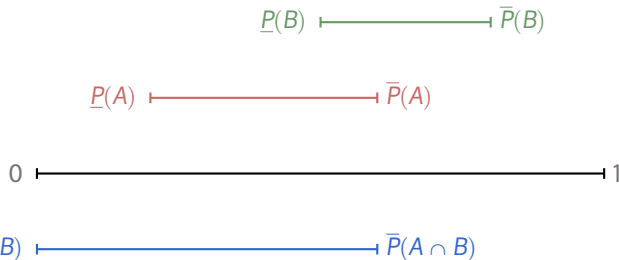
Maximise and minimise  $\beta$  under the constraints:

$$\alpha + \beta + \gamma + \delta = 1$$

$$\alpha, \beta, \gamma, \delta \geq 0$$

$$\underline{P}(A) \leq \alpha + \beta \leq \bar{P}(A)$$

$$\underline{P}(B) \leq \beta + \gamma \leq \bar{P}(B)$$



$$\underline{P}(A \cap B) = \max\{0, \underline{P}(A) + \underline{P}(B) - 1\} \text{ and } \bar{P}(A \cap B) = \min\{\bar{P}(A), \bar{P}(B)\}$$

# CREDAL SETS

# Sets of probability measures



**closed and convex**  
**set of probability measures**  $\mathcal{M}$  on  $\Omega$   
=  
CREDAL SET



# Sets of probability measures



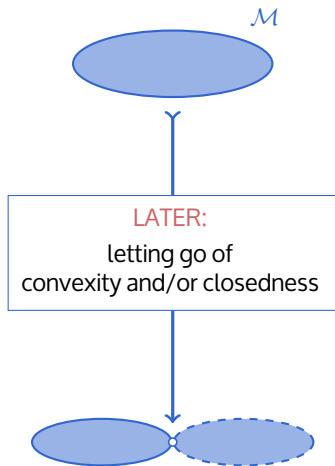
**closed and convex**  
**set of probability measures**  $\mathcal{M}$  on  $\Omega$   
=  
CREDAL SET



lower and upper envelopes of  $\mathcal{M}$ :

$$\underline{P}(C) := \min\{P(C) : P \in \mathcal{M}\} \text{ and } \bar{P}(C) := \max\{P(C) : P \in \mathcal{M}\}$$

# Sets of probability measures



$\Downarrow$

**closed and convex**  
**set of probability measures**  $\mathcal{M}$  on  $\Omega$   
=  
CREDAL SET

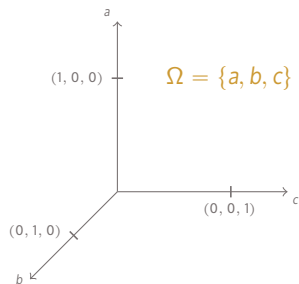
$\Downarrow$

lower and upper envelopes of  $\mathcal{M}$ :

$$\underline{P}(C) := \min\{P(C) : P \in \mathcal{M}\} \text{ and } \bar{P}(C) := \max\{P(C) : P \in \mathcal{M}\}$$

# EXPECTATION INTERVALS

# Expectation intervals



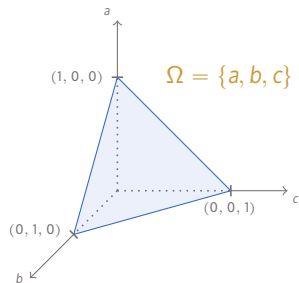
$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$

# Expectation intervals

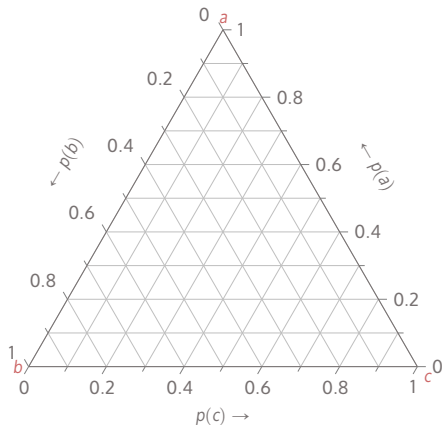


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

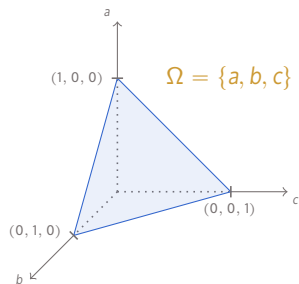
$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$



# Expectation intervals

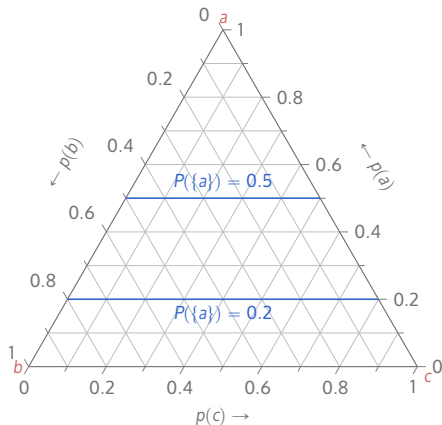


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

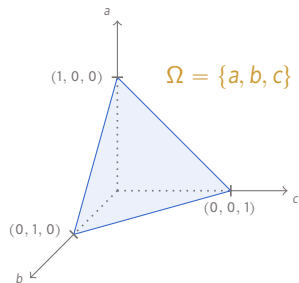
$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$



# Expectation intervals

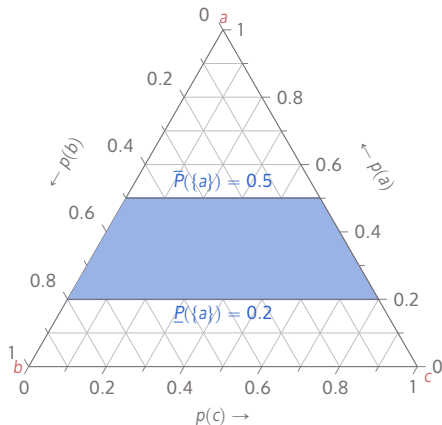


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

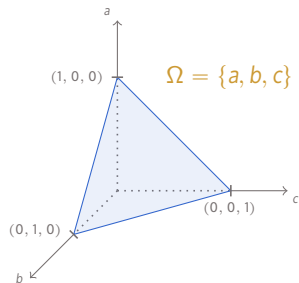
$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$



# Expectation intervals

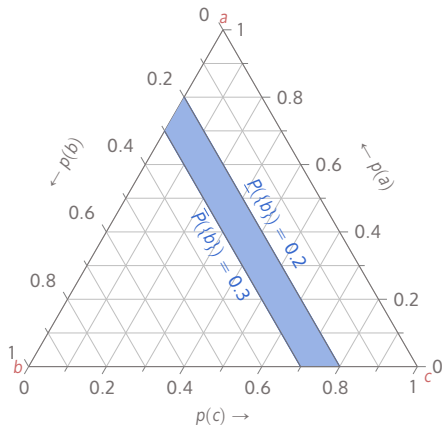


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

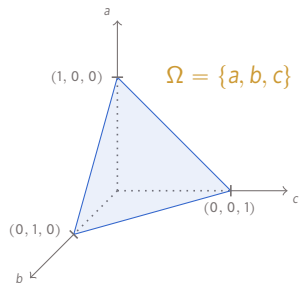
$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$





# Expectation intervals

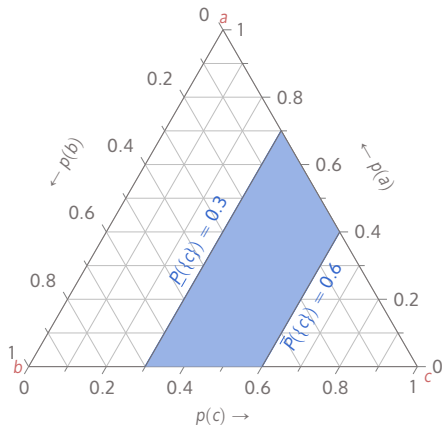


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

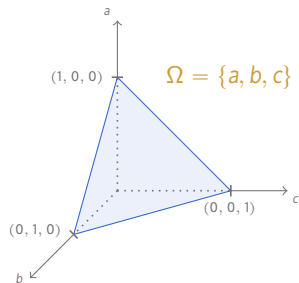
$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$



# Expectation intervals

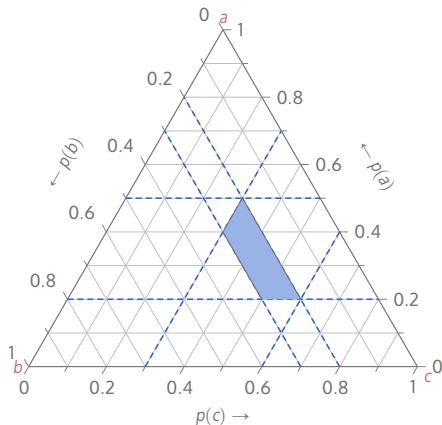


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

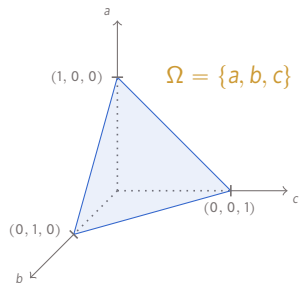
$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$



# Expectation intervals

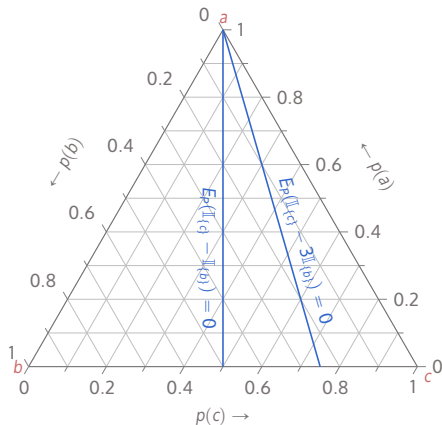


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

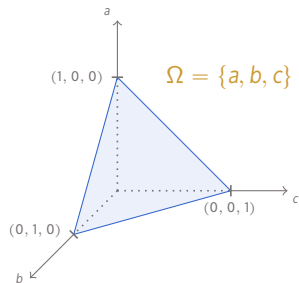
$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$



# Expectation intervals

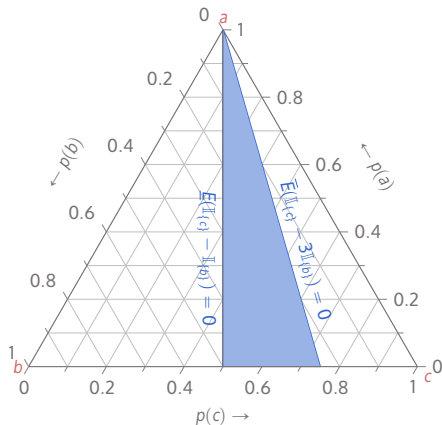


$$(1, 0, 0) \equiv \mathbb{I}_{\{a\}}$$

$$(0, 1, 0) \equiv \mathbb{I}_{\{b\}}$$

$$(0, 0, 1) \equiv \mathbb{I}_{\{c\}}$$

$$(\alpha, \beta, \gamma) \equiv \underbrace{\alpha \mathbb{I}_{\{a\}} + \beta \mathbb{I}_{\{b\}} + \gamma \mathbb{I}_{\{c\}}}_{\text{gamble } g: \Omega \rightarrow \mathbb{R}}$$



# Expectation intervals

assessment of  
lower and upper expectations  
of certain gambles

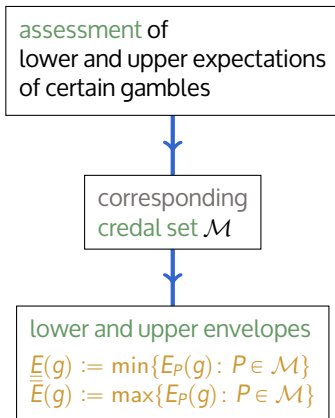
# Expectation intervals

assessment of  
lower and upper expectations  
of certain gambles

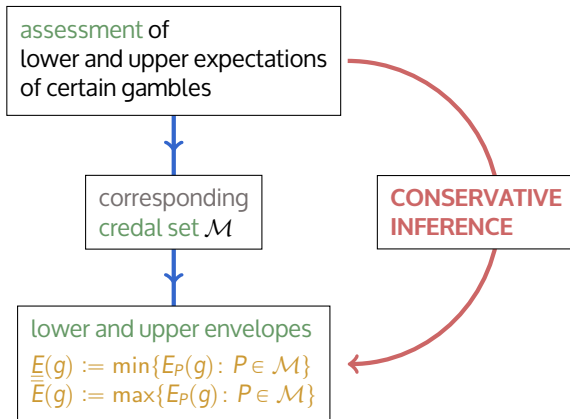


corresponding  
credal set  $\mathcal{M}$

# Expectation intervals



# Expectation intervals





# LOWER AND UPPER EXPECTATIONS

# Lower and upper expectations

## Lower expectation:

- +  $\inf f \leq \underline{E}(f)$
- +  $\underline{E}(f + g) \geq \underline{E}(f) + \underline{E}(g)$
- +  $\underline{E}(\lambda f) = \lambda \underline{E}(f)$  for  $\lambda \geq 0$

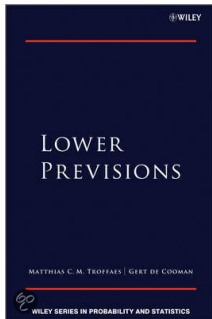
## Conjugacy:

- +  $\bar{E}(f) = -\underline{E}(-f)$

## Expectation:

- +  $\inf f \leq E(f)$
- +  $E(f + g) = E(f) + E(g)$
- +  $E(\lambda f) = \lambda E(f)$  for all  $\lambda$

# Lower and upper expectations



## Lower expectation:

- +  $\inf f \leq \underline{E}(f)$
- +  $\underline{E}(f + g) \geq \underline{E}(f) + \underline{E}(g)$
- +  $\underline{E}(\lambda f) = \lambda \underline{E}(f)$  for  $\lambda \geq 0$

## Conjugacy:

- +  $\bar{E}(f) = -\underline{E}(-f)$

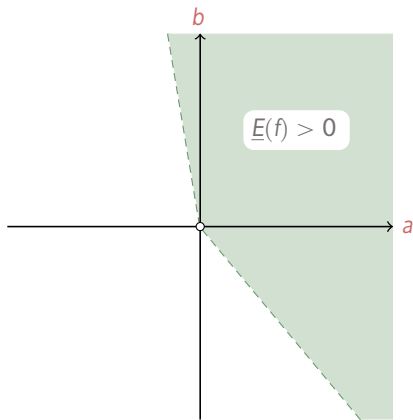
## Expectation:

- +  $\inf f \leq E(f)$
- +  $E(f + g) = E(f) + E(g)$
- +  $E(\lambda f) = \lambda E(f)$  for all  $\lambda$

# DECISION MAKING

# Decision making

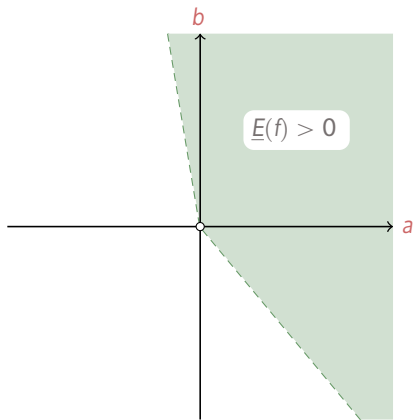
$$f > 0 \Leftrightarrow (\underline{E}(f) > 0 \text{ or } f > 0)$$



# Decision making

$$f > 0 \Leftrightarrow (\underline{E}(f) > 0 \text{ or } f > 0)$$

$$g > f \Leftrightarrow g - f > 0$$



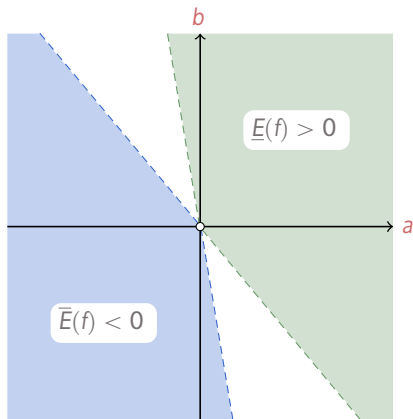
# Decision making

$$f > 0 \Leftrightarrow (\underline{E}(f) > 0 \text{ or } f > 0)$$

$$g > f \Leftrightarrow g - f > 0$$

$$0 > f \Leftrightarrow -f > 0$$

$$\Leftrightarrow (\bar{E}(f) < 0 \text{ or } f < 0)$$



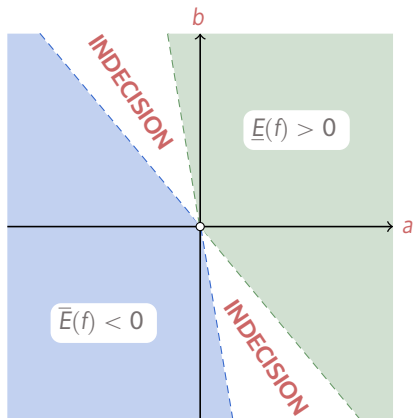
# Decision making

$$f > 0 \Leftrightarrow (\underline{E}(f) > 0 \text{ or } f > 0)$$

$$g > f \Leftrightarrow g - f > 0$$

$$0 > f \Leftrightarrow -f > 0$$

$$\Leftrightarrow (\bar{E}(f) < 0 \text{ or } f < 0)$$





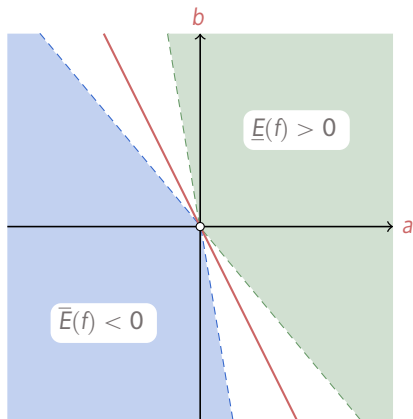
# Decision making

$$f > 0 \Leftrightarrow (\underline{E}(f) > 0 \text{ or } f > 0)$$

$$g > f \Leftrightarrow g - f > 0$$

$$0 > f \Leftrightarrow -f > 0$$

$$\Leftrightarrow (\bar{E}(f) < 0 \text{ or } f < 0)$$



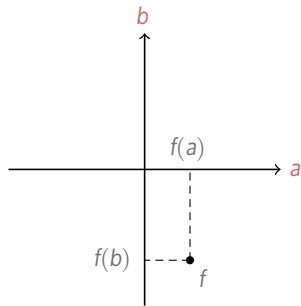
# DESIRABLE GAMBLES

# Binary choice: desirable gambles

A **gamble**  $f: \Omega \rightarrow \mathbb{R}$  is **desirable** if it is **strictly preferred** to the zero gamble 0—the status quo.

The **logic of desirability** is based on elementary statements

$\vdash_D f \longrightarrow$  'the gamble  $f$  is desirable'.



# Binary choice: desirable gambles

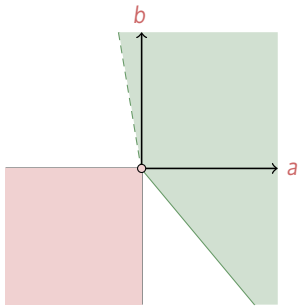
A **gamble**  $f: \Omega \rightarrow \mathbb{R}$  is **desirable** if it is **strictly preferred** to the zero gamble  $0$ —the status quo.

The **logic of desirability** is based on elementary statements

$$\vdash_D f \longrightarrow \text{'the gamble } f \text{ is desirable'}.$$

This logic is governed by the following **axioms**:

- +  $\not\vdash_D 0$
- + if  $f > 0$  then  $\vdash_D f$
- + if  $(\vdash_D f \text{ and } \vdash_D g)$  then  $\vdash_D (f + g)$
- + if  $\vdash_D f$  then  $\vdash_D (\lambda f)$  for all real  $\lambda > 0$



# Binary choice: desirable gambles

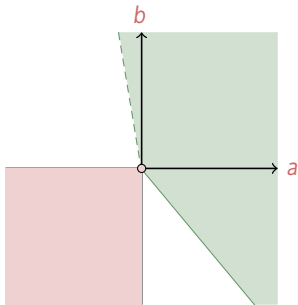
A **gamble**  $f: \Omega \rightarrow \mathbb{R}$  is **desirable** if it is **strictly preferred** to the zero gamble  $0$ —the status quo.

The **logic of desirability** is based on elementary statements

$$\vdash_D f \longrightarrow \text{'the gamble } f \text{ is desirable'}.$$

This logic is governed by the following **axioms**:

- +  $\not\vdash_D 0$
- + if  $f > 0$  then  $\vdash_D f$
- + if  $(\vdash_D f \text{ and } \vdash_D g)$  then  $\vdash_D (f + g)$
- + if  $\vdash_D f$  then  $\vdash_D (\lambda f)$  for all real  $\lambda > 0$



## CONSERVATIVE INFERENCE

# Binary choice: desirable gambles

The logic behind desirable gambles **underlies** all of (finitary) **probability theory**.

$$\underline{E}(f) := \sup\{\alpha \in \mathbb{R} : \vdash_D (f - \alpha)\}$$

# Binary choice: desirable gambles

The logic behind desirable gambles **underlies** all of (finitary) **probability theory**.

$$\underline{E}(f) := \sup\{\alpha \in \mathbb{R} : \vdash_D (f - \alpha)\}$$

**Bayes's rule** is part of this logic and is therefore **deductive**:

$$\underline{E}(f|A) := \sup\{\alpha \in \mathbb{R} : \vdash_D (f - \alpha)\mathbb{I}_A\}$$

# Binary choice: desirable gambles

ALLOWING FOR  
IMPRECISION LAYS BARE  
THE CONSERVATIVE  
INFERENCE MECHANISM  
BEHIND PROBABILISTIC  
REASONING

The logic behind desirable gambles **underlies** all of (finitary) probability theory.

$$\underline{E}(f) := \sup\{\alpha \in \mathbb{R} : \vdash_D (f - \alpha)\}$$

Bayes's rule is part of this logic and is therefore **deductive**:

$$\underline{E}(f|A) := \sup\{\alpha \in \mathbb{R} : \vdash_D (f - \alpha)\mathbb{I}_A\}$$





# CHOICE FUNCTIONS

# Non-binary choice: desirable sets of gambles

## RECALL: BINARY CHOICE

The logic of desirable gambles is governed by:

- +  $\not\vdash_D 0$
- + if  $f > 0$  then  $\vdash_D f$
- + if  $\vdash_D f, g$  then  $\vdash_D (f + g)$
- + if  $\vdash_D f$  then  $\vdash_D (\lambda f)$  for all real  $\lambda > 0$

# Non-binary choice: desirable sets of gambles

## RECALL: BINARY CHOICE

The logic of desirable gambles is governed by:

- +  $\not\vdash_D 0$
- + if  $f > 0$  then  $\vdash_D f$
- + if  $\vdash_D f, g$  then  $\vdash_D (f + g)$
- + if  $\vdash_D f$  then  $\vdash_D (\lambda f)$  for all real  $\lambda > 0$

We can get to **NON-BINARY CHOICE** and the theory of **choice functions** by (essentially) adding an **extra idea**:

- + if  $(\vdash_D f_1 \text{ or } \dots \text{ or } \vdash_D f_n)$  then  $\vdash_D \{f_1, \dots, f_n\}$

# Non-binary choice: desirable sets of gambles

## RECALL: BINARY CHOICE

The logic of desirable gambles is governed by:

- +  $\not\vdash_D 0$
- + if  $f > 0$  then  $\vdash_D f$
- + if  $\vdash_D f, g$  then  $\vdash_D (f + g)$
- + if  $\vdash_D f$  then  $\vdash_D (\lambda f)$  for all real  $\lambda > 0$

We can get to **NON-BINARY CHOICE** and the theory of **choice functions** by (essentially) adding an **extra idea**:

- + if  $(\vdash_D f_1 \text{ or } \dots \text{ or } \vdash_D f_n)$  then  $\vdash_D \{f_1, \dots, f_n\}$

## PROPOSITIONAL LOGIC ONLY WITH STATEMENTS:

' $f$  is desirable'

# Non-binary choice: desirable sets of gambles

## RECALL: BINARY CHOICE

The logic of desirable gambles is governed by:

- +  $\not\vdash_D 0$
- + if  $f > 0$  then  $\vdash_D f$
- + if  $\vdash_D f, g$  then  $\vdash_D (f + g)$
- + if  $\vdash_D f$  then  $\vdash_D (\lambda f)$  for all real  $\lambda > 0$

## REPRESENTATION RESULTS:

Levi's **E-admissibility** but with

- coherent set of desirable gambles instead of probability measure
- sets not necessarily closed nor convex
- extra axioms add extra structure

We can get to **NON-BINARY CHOICE** and the theory of **choice functions** by (essentially) adding an **extra idea**:

- + if  $(\vdash_D f_1 \text{ or } \dots \text{ or } \vdash_D f_n)$  then  $\vdash_D \{f_1, \dots, f_n\}$

## PROPOSITIONAL LOGIC ONLY WITH STATEMENTS:

**' $f$  is desirable'**

# Non-binary choice: desirable sets of gambles

ALLOWING FOR  
IMPRECISION LAYS BARE  
THE LINK BETWEEN  
PROBABILISTIC REASONING  
AND CHOICE THEORY



## RECALL: BINARY CHOICE

The logic of desirable gambles is governed by:

- +  $\not\vdash_D 0$
- + if  $f > 0$  then  $\vdash_D f$
- + if  $\vdash_D f, g$  then  $\vdash_D (f + g)$
- + if  $\vdash_D f$  then  $\vdash_D (\lambda f)$  for all real  $\lambda > 0$

We can get to **NON-BINARY CHOICE** and the theory of **choice functions** by (essentially) adding an **extra idea**:

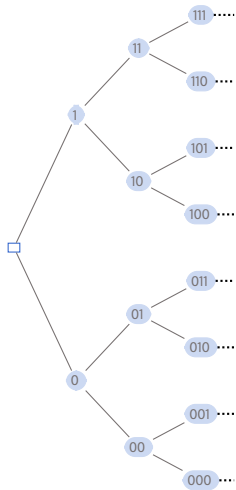
- + if  $(\vdash_D f_1 \text{ or } \dots \text{ or } \vdash_D f_n)$  then  $\vdash_D \{f_1, \dots, f_n\}$

**PROPOSITIONAL LOGIC ONLY WITH STATEMENTS:**

**' $f$  is desirable'**

# IMPRECISE STOCHASTIC PROCESSES

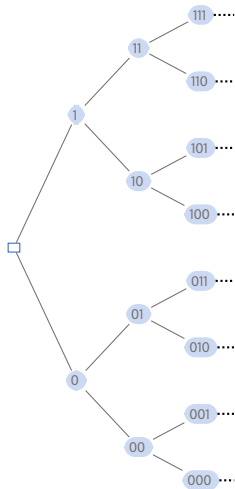
# Imprecision in stochastic processes



$X_1, X_2, X_3, \dots$



# Imprecision in stochastic processes

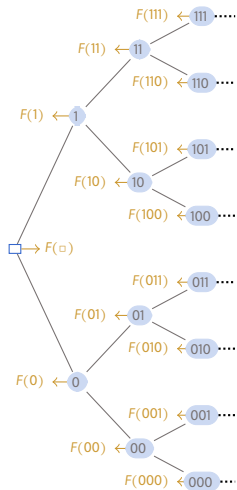


$X_1, X_2, X_3, \dots$

Situations  $s \in \mathbb{S}$  are the nodes in the event tree:  
finite strings of states

Paths  $\omega \in \Omega$  are the leaves in the event tree:  
infinite strings of states

# Imprecision in stochastic processes



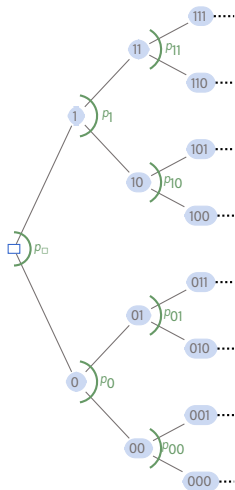
$X_1, X_2, X_3, \dots$

Situations  $s \in \mathbb{S}$  are the nodes in the event tree:  
finite strings of states

Paths  $\omega \in \Omega$  are the leaves in the event tree:  
infinite strings of states

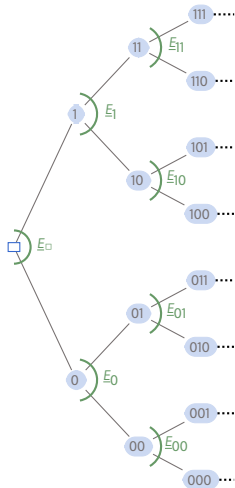
A process  $F: \mathbb{S} \rightarrow \mathbb{R}$  attaches a real number  $F(s)$  to every situation  $s$ .

# Imprecision in stochastic processes



A **precise probability tree** attaches a **local mass function**  $p_s$  to every situation  $s$ .

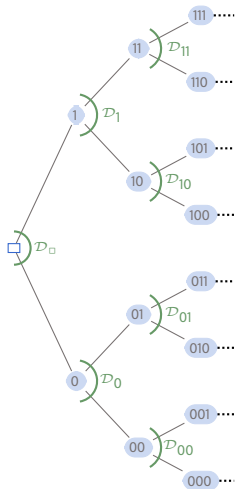
# Imprecision in stochastic processes



A **precise** probability tree attaches a **local mass function**  $p_s$  to every situation  $s$ .

An **imprecise** probability tree attaches a **local lower expectation**  $\underline{E}_s$  to every situation  $s$ .

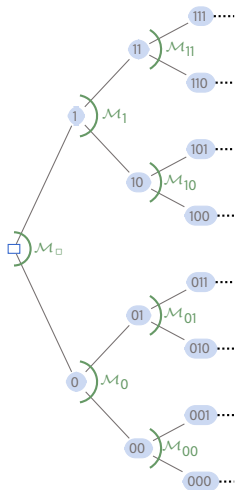
# Imprecision in stochastic processes



A **precise** probability tree attaches a **local mass function**  $p_s$  to every situation  $s$ .

An **imprecise** probability tree attaches a **local set of desirable gambles**  $\mathcal{D}_s$  to every situation  $s$ .

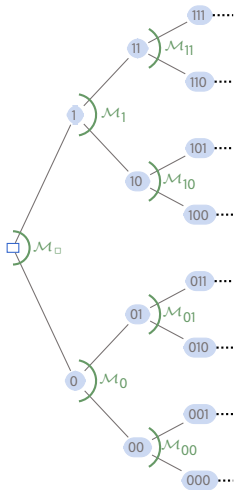
# Imprecision in stochastic processes



A **precise** probability tree attaches a **local mass function**  $p_s$  to every situation  $s$ .

An **imprecise** probability tree attaches a **local credal set**  $\mathcal{M}_s$  to every situation  $s$ .

# Imprecision in stochastic processes



A **precise probability tree** attaches a **local mass function**  $p_s$  to every situation  $s$ .

An **imprecise probability tree** attaches a **local credal set**  $\mathcal{M}_s$  to every situation  $s$ .

An imprecise probability tree is equivalent to a set of precise probability trees.

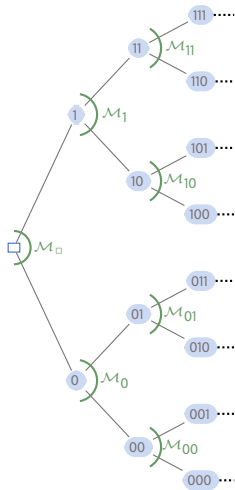
An imprecise probability tree is equivalent to a convex closed set of special processes, called supermartingales.



# IMPRECISE MARKOV CHAINS

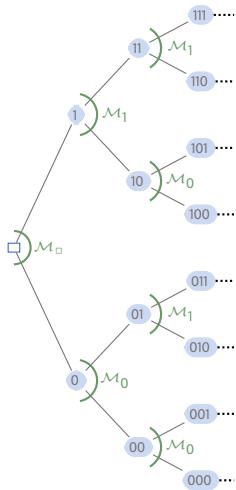


# Imprecision in Markov chains



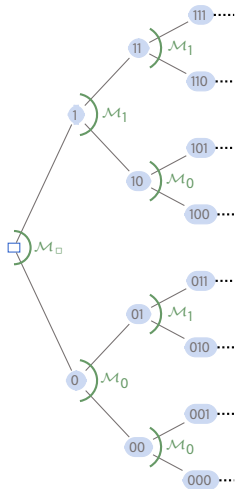
$$\mathcal{M}_{(x_1, \dots, x_n)} = \mathcal{M}_{x_n}$$

# Imprecision in Markov chains



$$\mathcal{M}_{(x_1, \dots, x_n)} = \mathcal{M}_{x_n}$$

# Imprecision in Markov chains

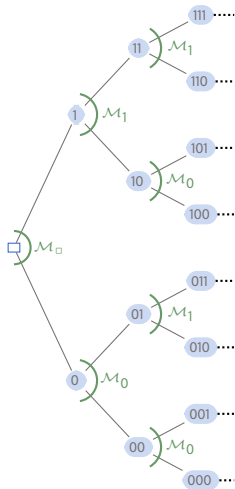


Due to the **Markov condition**

$$\mathcal{M}_{(x_1, \dots, x_n)} = \mathcal{M}_{x_n},$$

many inferences in **imprecise Markov chains** become **polynomial** in complexity, no longer exponential.

# Imprecision in Markov chains



Due to the **Markov condition**

$$\mathcal{M}_{(x_1, \dots, x_n)} = \mathcal{M}_{x_n},$$

many inferences in **imprecise Markov chains** become **polynomial** in complexity, no longer exponential.

**By allowing for imprecision, we can efficiently calculate conservative bounds on the behaviour of precise stochastic processes that are not Markov.**

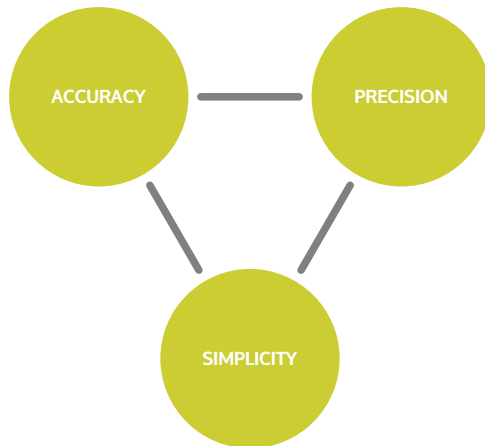
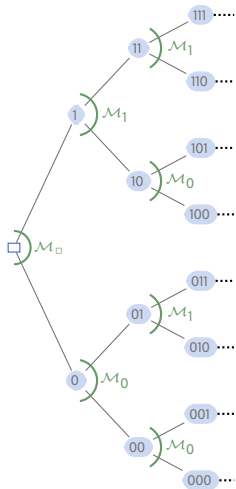


and



$\Rightarrow$  LUMPING

# Imprecision in Markov chains

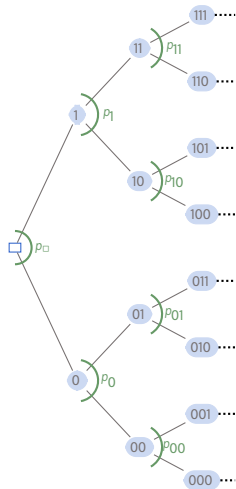


# ALGORITHMIC RANDOMNESS

# Imprecision in algorithmic randomness

011001101010111100101110001110101010011011011100011...

# Imprecision in algorithmic randomness



01100110101010111100101110001110101010011011011100011...

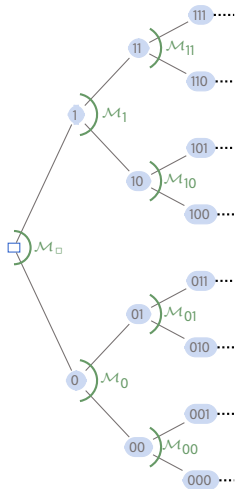
random for a **precise** probability tree

→ randomness tests

→ supermartingales



# Imprecision in algorithmic randomness



011001101010111100101110001110101010011011011100011...

random for a **precise** probability tree

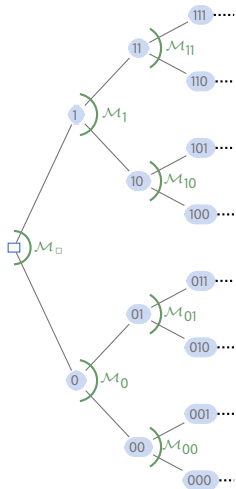
→ randomness tests  
→ supermartingales

random for an **imprecise** probability tree

→ randomness tests  
→ supermartingales



# Imprecision in algorithmic randomness

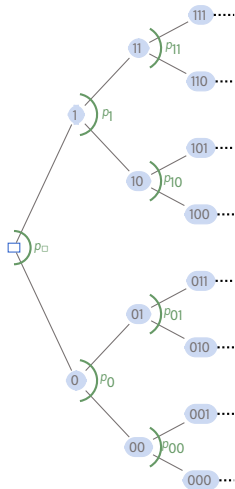


random for **more precise** probability tree



random for **less precise** probability tree

# Imprecision in algorithmic randomness

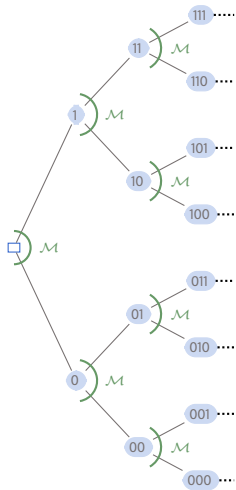


random for **more precise** probability tree

↓ but not ↑

random for **less precise** probability tree

# Imprecision in algorithmic randomness



random for **more precise** probability tree

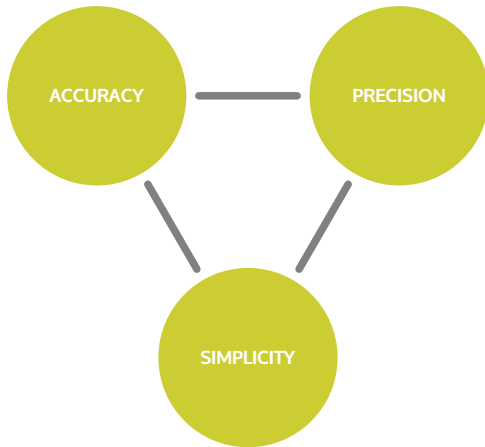
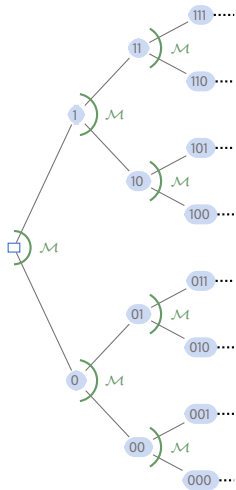
↓ but **not** ↑

random for **less precise** probability tree

**EXAMPLE:**

NONSTATIONARY PRECISE  $\Rightarrow$  STATIONARY IMPRECISE

# Imprecision in algorithmic randomness



**IN SUMMARY**

# In summary ...

## IMPRECISION IN PROBABILITY THEORY ALLOWS US AND HELPS US TO

- + deal honestly and systematically with indecision and incompleteness
- + see the precise special case in a much wider, structured mathematical perspective and context
- + identify and use the logic and conservative inference mechanisms behind probabilistic reasoning
- + provide a natural link between measure-theoretic and game-theoretic probability
- + look at and use simpler and more conservative models that are computationally more tractable

**THE END**