Optimal transport in Dempster-Shafer theory and its applications in game theory

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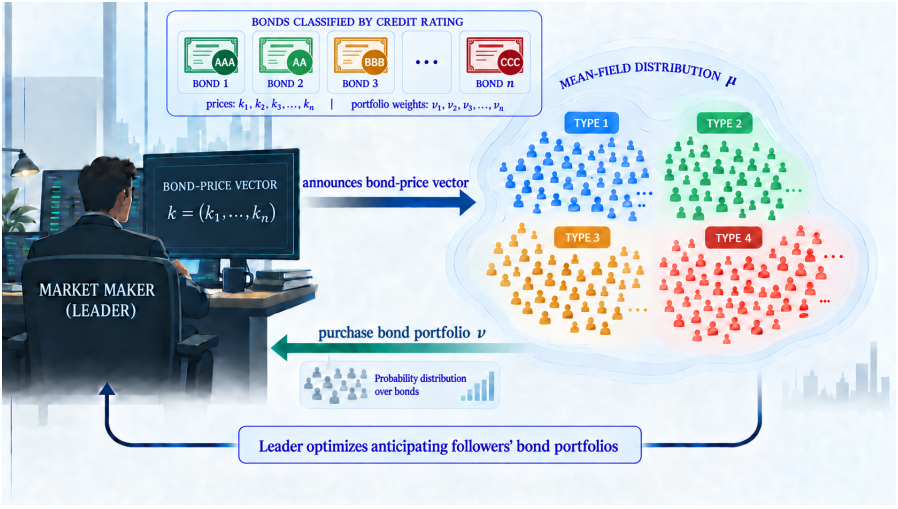


Overview of the seminar

- A motivating game theoretic example
- Optimal Transport in Dempster-Shafer theory and α-maxmin criterion
- Cournot-Nash games under DS uncertainty and α-maxmin preference
- Stackelberg-Cournot-Nash games under DS uncertainty and α-maxmin preference

Motivation

A motivating game theoretic example: A bond market (1)



A motivating game theoretic example: A bond market (2)

Mean-field interactions between investors: It does not matter who buys a bond type but rather how many investors buy it.

$\Longrightarrow$ **Congestion:** favoring dispersion of choices

$\Longrightarrow$ **Attraction:** favoring concentration of choices

Complete probabilistic knowledge: We need a complete knowledge of π.

Problem

What if we only have **partial knowledge** on the distribution π of investor types?

A motivating game theoretic example: A bond market (3)

Investor types x



x_1 = retired investor



x_2 = employee investor



x_3 = self-employed investor

Bond types y



y_1 = A-rated bond



y_2 = B-rated bond



y_3 = C-rated bond

Probabilistic information on investors

We know: $\geq 40\%$ investors are not retired and $\geq 60\%$ investors are not self-employed

$$\mathcal{P} = \{\pi \text{ probability measure } : \pi(\{x_2, x_3\}) \geq 0.4, \pi(\{x_1, x_2\}) \geq 0.6\}$$

generating a **belief** and a **plausibility functions**

$$\mu(A) = \min_{\pi \in \mathcal{P}} \pi(A) \quad \text{and} \quad \overline{\mu}(A) = \max_{\pi \in \mathcal{P}} \pi(A)$$

Existing literature on games with a continuum of players

Games with a **continuum of players**

- Aumann 1966
- Schemeidler 1973
- Mas-Colell 1984

$\Longrightarrow$ Only existence results

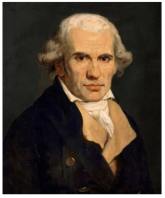
Probabilistic **optimal transport** (Villani 2009) formulation

- Blanchet et al. 2017 (Cournot-Nash game)
- Acciaio et al. 2025 (Stackelberg-Cournot-Nash game)

$\Longrightarrow$ Approximation of equilibria via entropic regularization

$\Longrightarrow$ **Not able to address partially known distributions**

Some history of optimal transport



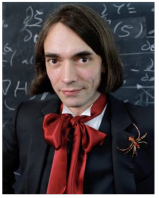
G. Monge
1781



L. Kantorovich
1942



L. Wasserstein
1970



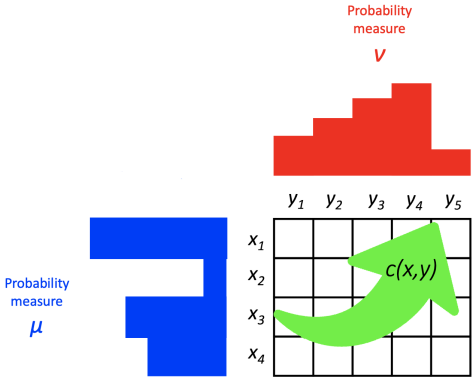
C. Villani
2009



G. Peyré
2015

TIME

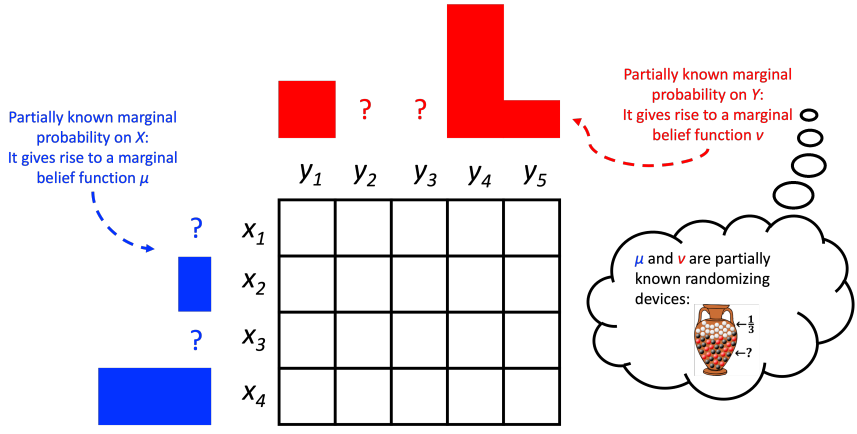
The classical optimal transport problem (Monge 1781; Kantorovich 1942)



Goal: Find a joint probability measure γ with marginals μ, ν, minimizing the expected cost $c(x, y)$.

$$\mathrm{OT}(\mu, \nu, c) = \min_{\gamma \in \mathbf{P}(\mu,\nu)} \mathbb{E}_{\gamma}[c]$$

The DS optimal transport problem



Dempster-Shafer theory

Dempster-Shafer theory

Let Ω be a finite non-empty set of states of the world.

Belief and plausibility functions (Dempster 1967; Shafer 1976)

A **belief function** $\mu : 2^{\Omega} \to [0,1]$ satisfies:

(i) $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$;

(ii) $\mu\left(\bigcup_{i=1}^{k} E_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} E_i\right)$, for all $k \geq 2$ and $E_1, \ldots, E_k \in 2^{\Omega}$.

Every ν is associated with a dual **plausibility function** defined as

$$\overline{\mu}(A) = 1 - \mu(A^c).$$

If *(ii)* holds as an equality, then $\mu = \overline{\mu}$ is a probability measure.

Möbius inverse

Every μ is associated with its **Möbius inverse** $m_\mu : 2^\Omega \to [0,1]$ s.t.

(i) $m_\mu(\emptyset) = 0$;

(ii) $\sum_{A \subseteq \Omega} m_\mu(A) = 1$.

$$\mathcal{F}_\mu = \{E \in 2^\Omega \,:\, m_\mu(E) > 0\} \text{ set of } \textbf{focal elements}$$

Bijection between μ's and m_μ's

$$\mu(A) = \sum_{B \in \mathcal{F}_\mu,\, B \subseteq A} m_\mu(B) \quad \text{and} \quad \overline{\mu}(A) = \sum_{B \in \mathcal{F}_\mu,\, B \cap A \neq \emptyset} m_\mu(B).$$

Why do we work with belief functions?

PROS:

1. They are the non-additive measures "closest" to probability measures in terms of properties
2. Every μ is the **lower envelope** of a **closed** (in $[0,1]^{2^{\Omega}}$ endowed with the product topology) and **convex** set of probability measures

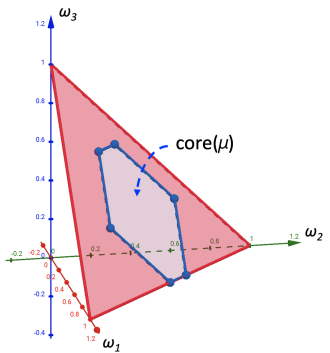
$$\textbf{core}(\mu) = \{\pi \,:\, \pi \text{ is a probability measure on } 2^{\Omega},\, \pi \geq \mu\}$$

CONS:

1. Their representation has generally exponential size in $d = |\Omega|$

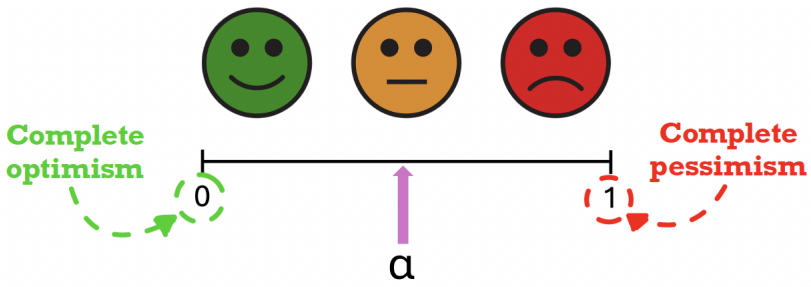
An ambiguous notion of expectation: The Choquet integral

- $\mathbb{C}_{\mu}[X] = \min_{\pi \in \mathbf{core}(\mu)} \mathbb{E}_{\pi}[X] = \sum_{B \in \mathcal{F}_{\mu}} \left(\min_{\omega \in B} X(\omega) \right) m_{\mu}(B)$
- $\mathbb{C}_{\overline{\mu}}[X] = -\mathbb{C}_{\mu}[-X]$
- $\mathbb{C}_{\overline{\mu}}[X] = \max_{\pi \in \mathbf{core}(\mu)} \mathbb{E}_{\pi}[X] = \sum_{B \in \mathcal{F}_{\mu}} \left(\max_{\omega \in B} X(\omega) \right) m_{\mu}(B)$
- If $\mu = \pi$ probability measure, then $\mathbb{C}_{\pi}[X] = \mathbb{E}_{\pi}[X]$



Hurwicz α-maxmin criterion (Hurwicz 1951)

We fix a **pessimism index** $\alpha \in [0, 1]$:



We map a random variable $X \in \mathbb{R}^{\Omega}$ to

$$\alpha \underbrace{\mathbb{C}_{\mu}[X]}_{\text{Lower expectation}} + (1-\alpha) \underbrace{\mathbb{C}_{\overline{\mu}}[X]}_{\text{Upper expectation}}$$

Literature on α-maxmin preferences

$$\underbrace{\alpha\left(\min_{\pi\in\mathbf{core}(\mu)}\mathbb{E}_{\pi}[X]\right)}_{\text{Lower expectation}}+(1-\alpha)\underbrace{\left(\max_{\pi\in\mathbf{core}(\mu)}\mathbb{E}_{\pi}[X]\right)}_{\text{Upper expectation}}$$

$$\|$$

$$\sum_{B\in\mathcal{F}_{\mu}}\left[\alpha\left(\min_{\omega\in B}X(\omega)\right)+(1-\alpha)\left(\max_{\omega\in B}X(\omega)\right)\right]m_{\mu}(B)$$

- Hurwicz 1951
- Ghirardato et al. 2004
- Denoeux 2019
- Chateauneuf et al. 2024

DS marginal problem

Marginal problem with belief functions

Consider:

- μ, belief function on $2^{\mathcal{X}}$, with $\mathcal{X} = \{x_1, \ldots, x_m\}$
- ν, belief function on $2^{\mathcal{Y}}$, with $\mathcal{Y} = \{y_1, \ldots, y_n\}$

DS marginal problem

Find a $\gamma \in \mathbf{B}(\mu, \nu)$, where $\mathbf{B}(\mu, \nu)$ is the set of belief function on $(\mathcal{X} \times \mathcal{Y}, 2^{\mathcal{X} \times \mathcal{Y}})$ with first marginal equal to μ and second marginal equal to ν.

$$\gamma \in \mathbf{B}(\mu, \nu) \iff \begin{cases} \sum_{D \subseteq A \times \mathcal{Y}} m_\gamma(D) = \mu(A), \ \forall A \in 2^{\mathcal{X}}, \\ \sum_{D \subseteq \mathcal{X} \times B} m_\gamma(D) = \nu(B), \ \forall B \in 2^{\mathcal{Y}}. \end{cases}$$

Complexity reduction of the DS marginal problem

Consider:

- μ, belief function on $2^{\mathcal{X}}$, with $\mathcal{F}_\mu = \{E_1, \ldots, E_M\}$
- ν, belief function on $2^{\mathcal{Y}}$, with $\mathcal{F}_\nu = \{F_1, \ldots, F_N\}$
- $\mathcal{F}_\mu \otimes \mathcal{F}_\nu := \{E_i \times F_j \,:\, E_i \in \mathcal{F}_\mu, F_j \in \mathcal{F}_\nu\}$

$$\overline{\mathbf{B}}(\mu,\nu) = \{\gamma \in \mathbf{B}(\mu,\nu) : \mathcal{F}_\gamma \subseteq \mathcal{F}_\mu \otimes \mathcal{F}_\nu\}$$

Proposition

The following statements hold:

(i) $\overline{\mathbf{B}}(\mu,\nu) \subseteq \mathbf{B}(\mu,\nu)$; ← *the inclusion is generally strict*

(ii) $\overline{\mathbf{B}}(\mu,\nu)$ *is a non-empty closed convex subset of* $[0,1]^{2^{\mathcal{X}\times\mathcal{Y}}}$, *endowed with the product topology;*

(iii) $\overline{\mathbf{B}}(\mu,\nu)$ *is determined by a finite set of extreme points* $\mathbf{ext}(\overline{\mathbf{B}}(\mu,\nu))$.

Joint Möbius inverses from copulas

Choose:

- A permutation σ of $\{1, \ldots, M\}$ s.t. $E_{\sigma(1)} \leq \cdots \leq E_{\sigma(M)}$
- A permutation τ of $\{1, \ldots, N\}$ s.t. $F_{\tau(1)} \leq \cdots \leq F_{\tau(N)}$
- A **copula** $\mathsf{C} : [0,1]^2 \to [0,1]$

Diagonal difference (Ferson et al. 2004; Malinosky et al. 2023)

Define $m^{\mu,\nu,\sigma,\tau,\mathsf{C}} : 2^{\mathcal{X}\times\mathcal{Y}} \to [0,1]$ for $E_{\sigma(i)} \times F_{\tau(j)} \in \mathcal{F}_\mu \otimes \mathcal{F}_\nu$:

$$\begin{aligned} m^{\mu,\nu,\sigma,\tau,\mathsf{C}}(E_{\sigma(i)} \times F_{\tau(j)}) &= \mathsf{C}\left(M_\mu^\sigma(i), M_\nu^\tau(j)\right) + \mathsf{C}\left(M_\mu^\sigma(i-1), M_\nu^\tau(j-1)\right) \\ &\quad - \mathsf{C}\left(M_\mu^\sigma(i), M_\nu^\tau(j-1)\right) - \mathsf{C}\left(M_\mu^\sigma(i-1), M_\nu^\tau(j)\right), \end{aligned}$$

where $M_\mu^\sigma(i) = \sum_{k\leq i} m_\mu(E_{\sigma(k)})$, $M_\nu^\tau(j) = \sum_{k\leq j} m_\nu(F_{\tau(k)})$, and $m^{\mu,\nu,\sigma,\tau,\mathsf{C}}$ is 0 elsewhere.

Structure of $\overline{\mathbf{B}}(\mu, \nu)$

Theorem (Analog of Sklar's theorem)

Let σ and τ be permutations of $\{1, \dots, M\}$ and $\{1, \dots, N\}$, respectively. It holds that $\gamma \in \overline{\mathbf{B}}(\mu, \nu)$ if and only if there exists a bivariate copula C such that $m_\gamma = m^{\mu,\nu,\sigma,\tau,\mathsf{C}}$.

Unfortunately, we cannot characterize $\mathbf{ext}(\overline{\mathbf{B}}(\mu, \nu))$ by means of extreme copulas by varying σ, τ:

- $\mathsf{C}_L(x, y) := \max\{x + y - 1, 0\}$
- $\mathsf{C}_M(x, y) := \min\{x, y\}$
- $\mathsf{C}_L(x, y) \leq \mathsf{C}(x, y) \leq \mathsf{C}_M(x, y)$, for all copula C

$\Longrightarrow$ DSOT^{α} can be expressed in terms of the set of all copulas $\mathfrak{C}$

α-DS optimal transport problem

The α-DS optimal transport problem

Consider:

- μ, belief function on $2^{\mathcal{X}}$, with $\mathcal{X} = \{x_1, \ldots, x_m\}$;
- ν, belief function on $2^{\mathcal{Y}}$, with $\mathcal{Y} = \{y_1, \ldots, y_n\}$;
- $c : \mathcal{X} \times \mathcal{Y} \to [0, +\infty)$, a cost function;
- $\alpha \in [0, 1]$, **pessimism index (à la Hurwicz)**.

α-DS optimal transport problem (Lorenzini et al. 2026)

$$\mathrm{DSOT}^{\alpha}(\mu, \nu, c) = \min_{\gamma \in \mathbf{B}(\mu,\nu)} \{\alpha \mathbb{C}_{\gamma}[c] + (1 - \alpha)\mathbb{C}_{\overline{\gamma}}[c]\}.$$

$\Longrightarrow$ Other approaches: Gal et al. 2019, Bronevich et al. 2021, Torra 2023, Caprio 2025

Linear expression of DSOT^{α}

For every $E \times F \in \mathcal{F}_\gamma \subseteq \mathcal{F}_\mu \otimes \mathcal{F}_\nu$:

- $C(E \times F) = \min_{(x,y)\in E\times F} c(x,y)$
- $\overline{C}(E \times F) = \max_{(x,y)\in E\times F} c(x,y)$

$$\mathrm{DSOT}^{\alpha}(\mu,\nu,c) = \min_{\gamma\in\overline{\mathbf{B}}(\mu,\nu)} \sum_{E\times F\in\mathcal{F}_\gamma} (\alpha C(E\times F) + (1-\alpha)\overline{C}(E\times F))m_\gamma(E\times F).$$



It is a **linear programming problem** on the space of Möbius inverses of $\overline{\mathbf{B}}(\mu,\nu)$

Choquet-Wasserstein pseudo-distances (1)

Consider the **particular case**:

- $\mathcal{X} = \mathcal{Y} = \{x_1, \ldots, x_n\}$
- $c : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ is a **metric** on $\mathcal{X}$
- $c^p(x, y) := (c(x, y))^p$, for all $x, y \in \mathcal{X}$ and $p \geq 1$

Definition (Choquet-Wasserstein pseudo-distance of order $p \geq 1$)

For all belief function μ, ν on $2^{\mathcal{X}}$:

$$\begin{aligned} d_{\mathcal{CW}}^{p,\alpha}(\mu, \nu) &= (\mathrm{DSOT}^{\alpha}(\mu, \nu, c^p))^{\frac{1}{p}} \\ &= \left(\min_{\gamma \in \overline{\mathbf{B}}(\mu,\nu)} \{\alpha \mathbb{C}_{\gamma}[c^p] + (1-\alpha)\mathbb{C}_{\overline{\gamma}}[c^p]\} \right)^{\frac{1}{p}} \end{aligned}$$

Choquet-Wasserstein pseudo-distances (2)

$$\begin{array}{rcl} \mathbf{Bel}(\mathcal{X}) & := & \{\mu : \mu \text{ is a belief function on } 2^{\mathcal{X}}\} \\ \bigcup\mid & & \\ \mathbf{Pro}(\mathcal{X}) & := & \{\mu : \mu \text{ is a probability measure on } 2^{\mathcal{X}}\} \end{array}$$

The function $d_{\mathcal{CW}}^{p,\alpha}$ as the following properties

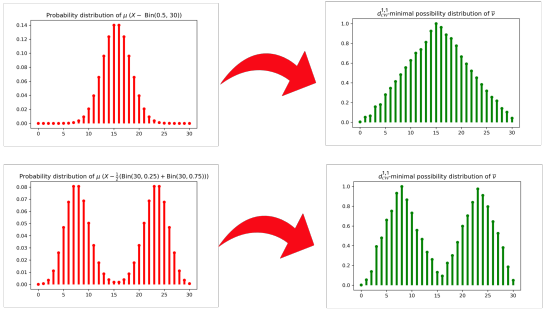
- $d_{\mathcal{CW}}^{p,\alpha}$ is a **metric** on $\mathbf{Pro}(\mathcal{X})$
- $d_{\mathcal{CW}}^{p,\alpha}$ is **not a metric** on $\mathbf{Bel}(\mathcal{X})$
- For $\alpha = 1$, $d_{\mathcal{CW}}^{p,1}$ is a **dissimilarity function** on $\mathbf{Bel}(\mathcal{X})$
- For $\alpha = 0$, $d_{\mathcal{CW}}^{p,0}$ is a **metric-like function** on $\mathbf{Bel}(\mathcal{X})$

$\Longrightarrow$ Comparison and approximation: Destercke et al. 2019, Montes et al. 2019

Choquet-Wasserstein pseudo-distances (3)

- $\mu \in \mathbf{Bel}(\mathcal{X})$
- $\mathcal{B} \subseteq \mathbf{Bel}(\mathcal{X})$, closed and convex

$$\nu^* \in \arg\min_{\nu \in \mathcal{B}} d_{\mathcal{CW}}^{\mathcal{P},\alpha}(\mu, \nu)$$



Cournot-Nash game under DS uncertainty and α-maxmin preferences

α-DS Cournot-Nash game (1)



Infinite population of players

- $\mathcal{X} = \{x_1, \ldots, x_m\}$, **player types**
- $\mathcal{Y} = \{y_1, \ldots, y_n\}$, **actions**
- $\mu \in \mathbf{Bel}(\mathcal{X})$ with $\mathcal{F}_\mu = \{E_1, \ldots, E_M\}$, **"imprecise" player type distribution**
- $\mathbf{Pro}(\mathcal{Y})$, set of **action distributions** ($\nu \in \mathbf{Pro}(\mathcal{Y})$ iff $\mathcal{F}_\nu \subseteq \{\{y_1\}, \ldots, \{y_n\}\}$)
- $\overline{\mathbf{B}}(\mu, \cdot)$ joint belief functions γ's with marginal μ and $\mathcal{F}_\gamma \subseteq \mathcal{F}_\mu \otimes \{\{y_1\}, \ldots, \{y_n\}\}$
- $k \in K \subseteq \mathbb{R}^{\mathcal{Y}}$, exogenous vector $k = (k(y_1), \ldots, k(y_n))$ of **action prices**

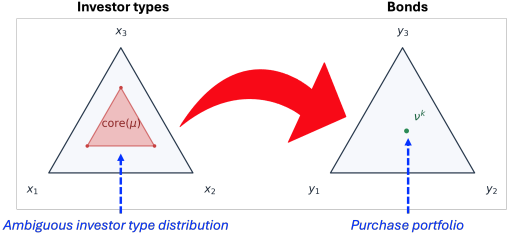
α-DS Cournot-Nash game (2)



Question

What is an equilibrium in this game?

For an exogenously given bond price vector $k = (k_1, k_2, k_3)$:



Players cost function

PLAYERS COST FUNCTION: $c^{\nu,k} : \mathcal{X} \times \mathcal{Y} \to [0, +\infty)$

parametric in $\nu \in \mathbf{Pro}(\mathcal{Y})$ and $k \in \mathbb{R}^{\mathcal{Y}}$, in the form

$$c^{\nu,k}(x,y) = \underbrace{c(x,y)}_{\substack{\text{cost for type } x \\ \text{of action } y}} + \underbrace{k(y)}_{\substack{\text{exogenous} \\ \text{action price}}} + \underbrace{f_y(\nu(y))}_{\substack{\text{penalty for} \\ \text{choosing } y}} + \underbrace{\sum_{\overline{y} \in \mathcal{Y}} \theta(\overline{y}, y)\nu(\overline{y})}_{\substack{\text{interactions with} \\ \text{other choices}}},$$

where

- f_y are **non-decreasing and continuous functions**
- θ is **symmetric**

α-DS Cournot-Nash equilibrium

Consider the **investors-only game** with exogenous action prices vector $k \in K$:

α-DS Cournot-Nash equilibrium (α-DS-CNE)

A $\gamma^k \in \overline{\mathbf{B}}(\mu, \cdot)$ with **additive** $\mathcal{Y}$-marginal ν^k is an α-**DS-CNE** w.r.t. $k \in K$ if it solves

$$\min_{\gamma \in \overline{\mathbf{B}}(\mu,\cdot)} \Big\{ \alpha \mathbb{C}_{\gamma}[c^{\nu^k,k}] + (1-\alpha)\mathbb{C}_{\overline{\gamma}}[c^{\nu^k,k}] \Big\}.$$

$\Longrightarrow$ To assure additivity of the $\mathcal{Y}$-marginal we consider $\gamma \in \overline{\mathbf{B}}(\mu, \cdot)$ with

$$\mathcal{F}_{\gamma} \subseteq \mathcal{F}_{\mu} \otimes \{\{y_1\}, \ldots, \{y_n\}\}$$

Finding α-DS-CNEs: Variational formulation

Energy functional on $\mathbf{Pro}(\mathcal{Y})$

$$\mathcal{E}(\nu) = \sum_{y \in \mathcal{Y}} \int_0^{\nu(y)} f_y(s)\mathrm{d}s + \frac{1}{2} \sum_{y, \overline{y} \in \mathcal{Y}} \theta(\overline{y}, y)\nu(\overline{y})\nu(y)$$

Theorem

Assume $\mathcal{E}$ is convex. Then, α-DS-CNEs exit and $\gamma^k \in \overline{\mathbf{B}}(\mu, \cdot)$ with $\mathcal{Y}$-marginal ν^k is an α-DS-CNE if and only if it is an optimizer for $\mathrm{DSOT}^{\alpha}(\mu, \nu^k, c)$ and it solves

$$\min_{\nu \in \mathbf{Pro}(\mathcal{Y})} \left\{ \mathrm{DSOT}^{\alpha}(\mu, \nu, c) + \sum_{y \in \mathcal{Y}} k(y)\nu(y) + \mathcal{E}(\nu) \right\}.$$

Finding α-DS-CNEs: Computational issue

$$\nu^k \text{ solves } \min_{\nu \in \mathbf{Pro}(\mathcal{Y})} \left\{ \mathrm{DSOT}^{\alpha}(\mu, \nu, c) + \sum_{y \in \mathcal{Y}} k(y)\nu(y) + \mathcal{E}(\nu) \right\}$$

$$\Updownarrow$$

$$k \in -\left(\underbrace{\partial_{\nu}[\mathrm{DSOT}^{\alpha}(\mu, \nu^k, c)]}_{\text{Subdifferential}} + \nabla \mathcal{E}(\nu^k) \right)$$

Computational issue

To find an α-DS-CNE we need to compute the subdifferential which is difficult.

Approximation via entropic regularization

Entropic regularization, for $\lambda > 0$

(Negative) DS-entropy: $H(\gamma) = \sum_{E\times\{y\}\in\mathcal{F}_\gamma} (\ln(m_\gamma(E\times\{y\})) - 1))m_\gamma(E\times\{y\})$

Ent. Reg.: $\mathrm{DSOT}^{\alpha}_{\lambda}(\mu,\nu,c) = \min_{\gamma\in\overline{\mathbf{B}}(\mu,\nu)} \left\{\alpha\mathbb{C}_\gamma[c^{\nu^k},k] + (1-\alpha)\mathbb{C}_{\overline{\gamma}}[c^{\nu^k},k] + \lambda H(\gamma)\right\}$

ADVANTAGES:

- it admits a unique optimizer γ^λ, that, for $\lambda \to 0$, satisfies
$$\gamma^\lambda \to \gamma^* := \arg\min\{H(\gamma) : \gamma \text{ optimizer of } \mathrm{DSOT}^{\alpha}(\mu,\nu,c)\};$$
- **Adapted Dykstra's algorithm:** To find an approximated α-DS-CNE w.r.t. k

Bond market example: α-DS Cournot-Nash equilibria (1)

Investor types

x



x_1 = retired investor



x_2 = employee investor



x_3 = self-employed investor

Bond types

y



y_1 = A-rated bond



y_2 = B-rated bond



y_3 = C-rated bond

- **Info:** $\geq 40\%$ investors are not retired and $\geq 60\%$ are not self-employed

$$m_\mu(\{x_1, x_2\}) = 0.6, \quad m_\mu(\{x_2, x_3\}) = 0.4$$

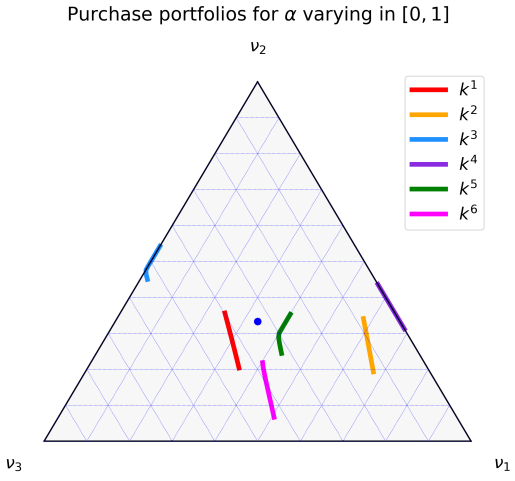
- **Investors cost parameters:**

$$c = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad f_y(\nu(y)) = \nu(y)^2, \quad \theta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

- **Exogenous bond prices vector:** $k = (2, 1, 0.5)$

Bond market example: α-DS Cournot-Nash equilibria (2)

$\mathcal{Y}$	y_1	y_2	y_3
k^1	3	2	1
k^2	1	2	3
k^3	4	1	0.5
k^4	0.5	1	4
k^5	1	1	1
k^6	1	2	0.5



Stackelberg-Cournot game under DS uncertainty and α-maxmin preferences

α-DS Stackelberg-Cournot-Nash game (1)



versus



Infinite population of players

- $\mathcal{X} = \{x_1, \ldots, x_m\}$, **player types**
- $\mathcal{Y} = \{y_1, \ldots, y_n\}$, **actions**
- $\mu \in \mathbf{Bel}(\mathcal{X})$ with $\mathcal{F}_\mu = \{E_1, \ldots, E_M\}$, **"imprecise" player type distribution**
- $\mathbf{Pro}(\mathcal{Y})$, set of **action distributions** ($\nu \in \mathbf{Pro}(\mathcal{Y})$ iff $\mathcal{F}_\nu \subseteq \{\{y_1\}, \ldots, \{y_n\}\}$)
- $\overline{\mathbf{B}}(\mu, \cdot)$ joint belief functions γ's with marginal μ and $\mathcal{F}_\gamma \subseteq \mathcal{F}_\mu \otimes \{\{y_1\}, \ldots, \{y_n\}\}$
- $K \subseteq \mathbb{R}^{\mathcal{Y}}$, set of **action prices** vectors $k = (k(y_1), \ldots, k(y_n))$ chosen by the **leader**

α-DS Stackelberg-Cournot-Nash game (2)

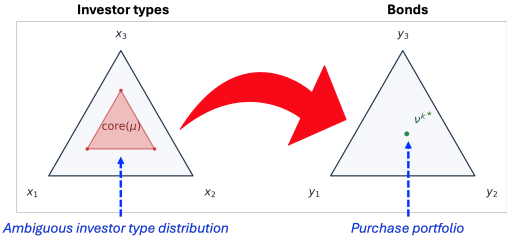


Question

What is an equilibrium in this game?

1) **The market maker chooses the bond price vector $k^* = (k_1^*, k_2^*, k_3^*)$:**

2)



Players and leader cost functions

PLAYERS COST FUNCTION: $c^{\nu,k} : \mathcal{X} \times \mathcal{Y} \to [0, +\infty)$

parametric in $\nu \in \mathbf{Pro}(\mathcal{Y})$ and $k \in \mathbb{R}^{\mathcal{Y}}$, in the form

$$c^{\nu,k}(x,y) = \underbrace{c(x,y)}_{\substack{\text{cost for type } x \\ \text{of action } y}} + \underbrace{k(y)}_{\substack{\text{exogenous} \\ \text{action price}}} + \underbrace{f_y(\nu(y))}_{\substack{\text{penalty for} \\ \text{choosing } y}} + \underbrace{\sum_{\overline{y} \in \mathcal{Y}} \theta(\overline{y}, y)\nu(\overline{y})}_{\substack{\text{interactions with} \\ \text{other choices}}},$$

where

- f_y are **non-decreasing and continuous functions**
- θ is **symmetric**

LEADER COST FUNCTION: $G : \mathbf{Pro}(\mathcal{Y}) \times K \to [0, +\infty)$, is **continuous**

α-DS Stackelberg-Cournot-Nash equilibrium

Consider the **leader-players game**:

α-DS Stackelberg-Cournot-Nash equilibrium (α-DS-SCNE)

A pair $(\gamma^{k^*}, k^*) \in \overline{\mathbf{B}}(\mu, \cdot) \times K$ is an α-**DS-SCNE** if it satisfies:

(i) γ^{k^*} is a α-DS-CNE w.r.t. k^*;

(ii) $G(\nu^{k^*}, k^*) \leq G(\nu^k, k)$ for all $k \in K$ and all γ^k α-DS-CNE w.r.t. k.

Theorem (Existence of α-DS-SCNEs)

If θ is positive semidefinite and K is compact then an α-DS-SCNE exists.

Finding equilibria: A distinguished subclass of problems

- Finding an α-DS-SCNE is computationally difficult for a general $G(\nu, k)$;
- The problem simplifies when $G(\nu, k) = G(\nu)$.

Finding an α-DS-SCNE when $G(\nu, k) = G(\nu)$

1) Find a minimizer ν^* of $G(\nu)$;
2) Find an optimizer γ^* of $\mathrm{DSOT}^{\alpha}(\mu, \nu^*, c)$;
3) Find $k^* \in (-\partial_\nu[\mathrm{DSOT}^{\alpha}(\mu, \nu^*, c)] - \nabla\mathcal{E}[\nu^*]) \cap K$



Approximation via entropic regularization:
Adapted Sinkhorn's algorithm

Bond market example: α-DS Stackelberg-Cournot-Nash equilibria (1)

Investor types

x



x_1 = retired investor



x_2 = employee investor



x_3 = self-employed investor

Bond types

y



y_1 = A-rated bond



y_2 = B-rated bond



y_3 = C-rated bond

- **Info:** $\geq 40\%$ investors are not retired and $\geq 60\%$ are not self-employed

$$m_\mu(\{x_1, x_2\}) = 0.6, \quad m_\mu(\{x_2, x_3\}) = 0.4$$

- **Investors cost parameters:**

$$c = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad f_y(\nu(y)) = \nu(y)^2, \quad \theta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

- **Market maker's set of bond prices vectors:** $K = [0, 100]^{\mathcal{Y}}$

Bond market example: α-DS Stackelberg-Cournot-Nash equilibria (2)

- **Market maker's cost function:** It favors diversification of the purchase portfolio
$$G(\nu, k) = \sum_{y \in \mathcal{Y}} \nu(y)^2$$
- **Unique minimizer of market maker's cost function for all k:** $\nu^* \equiv \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
- **Equilibrium price vectors**:

$\alpha \setminus \mathcal{Y}$	y_1	y_2	y_3
0	2.3292	1.6673	**1.0000**
0.1	2.3292	1.5673	**1.0000**
0.2	2.3292	1.4673	**1.0000**
0.3	2.3292	1.3673	**1.0000**
0.4	2.3292	1.2673	**1.0000**
0.5	2.3292	1.1673	**1.0000**
0.6	2.3292	1.0673	**1.0000**
0.7	2.3618	**1.0000**	1.0327
0.8	2.4618	**1.0000**	1.1327
0.9	2.5618	**1.0000**	1.2327
1	2.6618	**1.0000**	1.3327

Conclusions

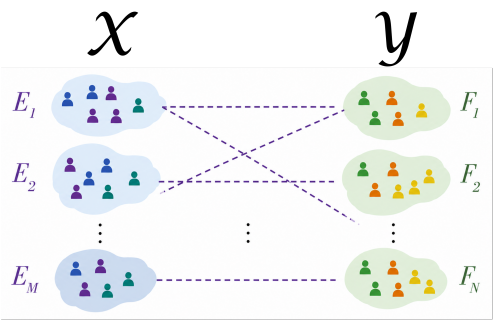
RECAP:

- **DSOT**$^{\alpha}$ models situations with partially specified marginal knowledge and α-maxmin preferences under ambiguity;
- α-**DS-CNEs** and α-**DS-SCNEs** in a game under ambiguity generalize results by Blanchet et al. (2017) and Acciaio et al. (2023);
- Adaptations of **Dykstra and Sinkhorn's algorithms** allow to find α-DS-CNEs and α-DS-SCNEs efficiently.

FUTURE PLANS:

- Extend the formulation to ambiguous marginals on $\mathcal{Y}$;
- Deal with more general cost functions leading to non-convex cases;
- Extend the model to infinite sets $\mathcal{X}$ and $\mathcal{Y}$.

What's next? Matching problem under ambiguity!



Sparsity of the imprecise transport plan (Cinfrignini er al. 2026)

- Quadratic regularization $\Longrightarrow$ global sparsity of m_γ
- Group Lasso regularization $\Longrightarrow$ localized sparsity of m_γ


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Thank you!

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