On the Interplay of Optimal Transport and Distributionally Robust Optimization

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NCCR Automation

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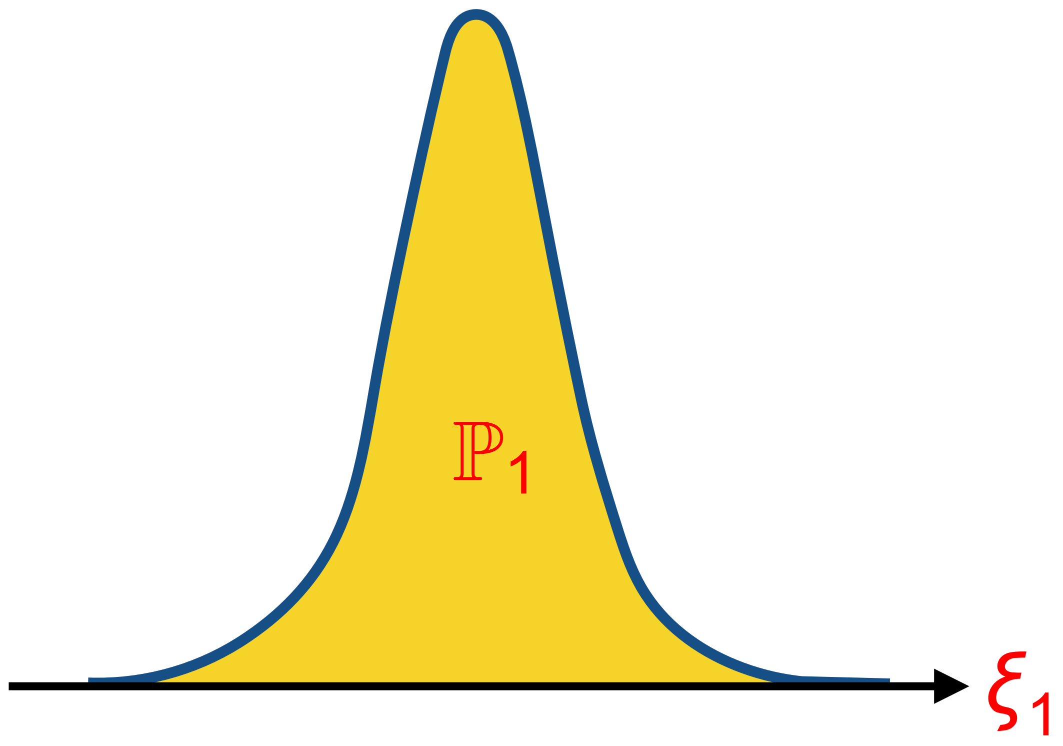
Optimal Transport (OT)

Pushforward Distribution



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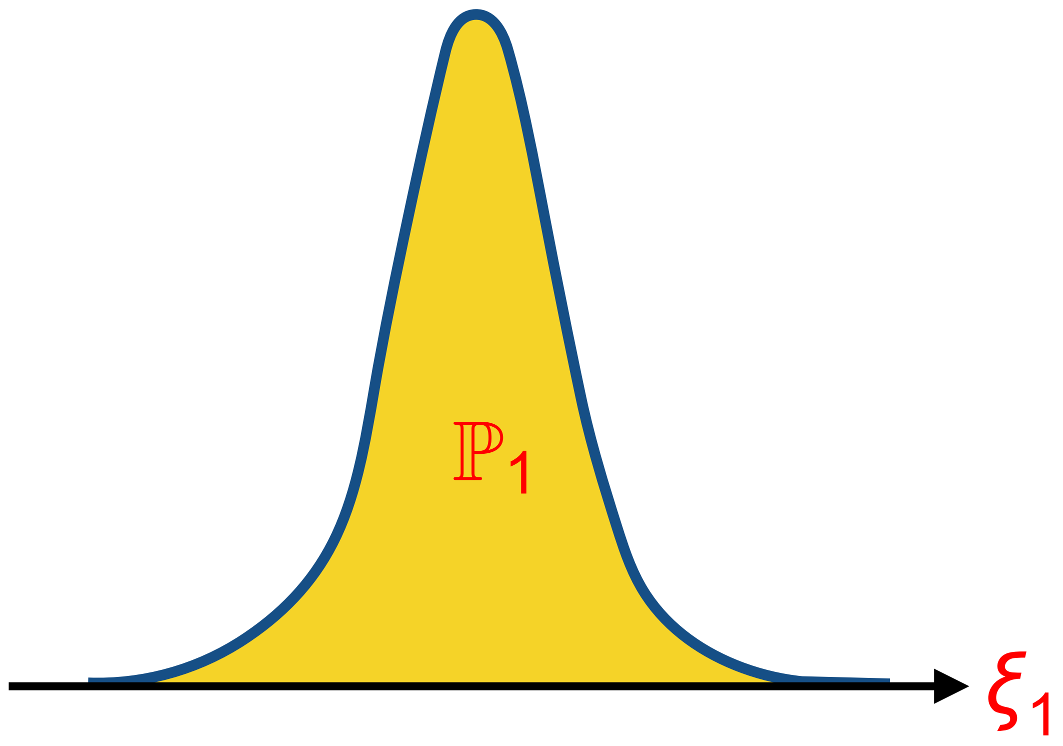
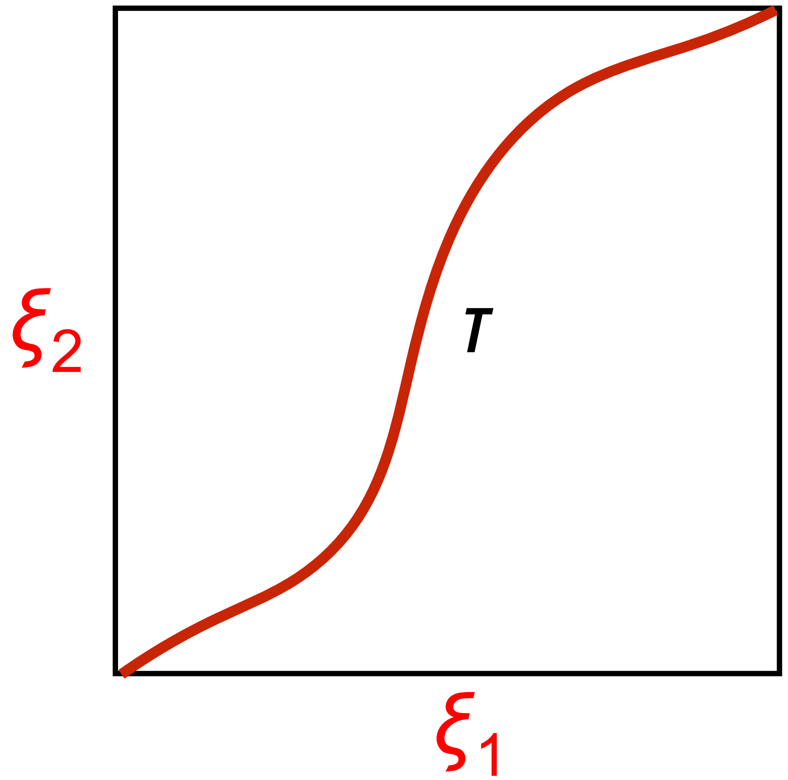


$$\xi_2 = T(\xi_1)$$



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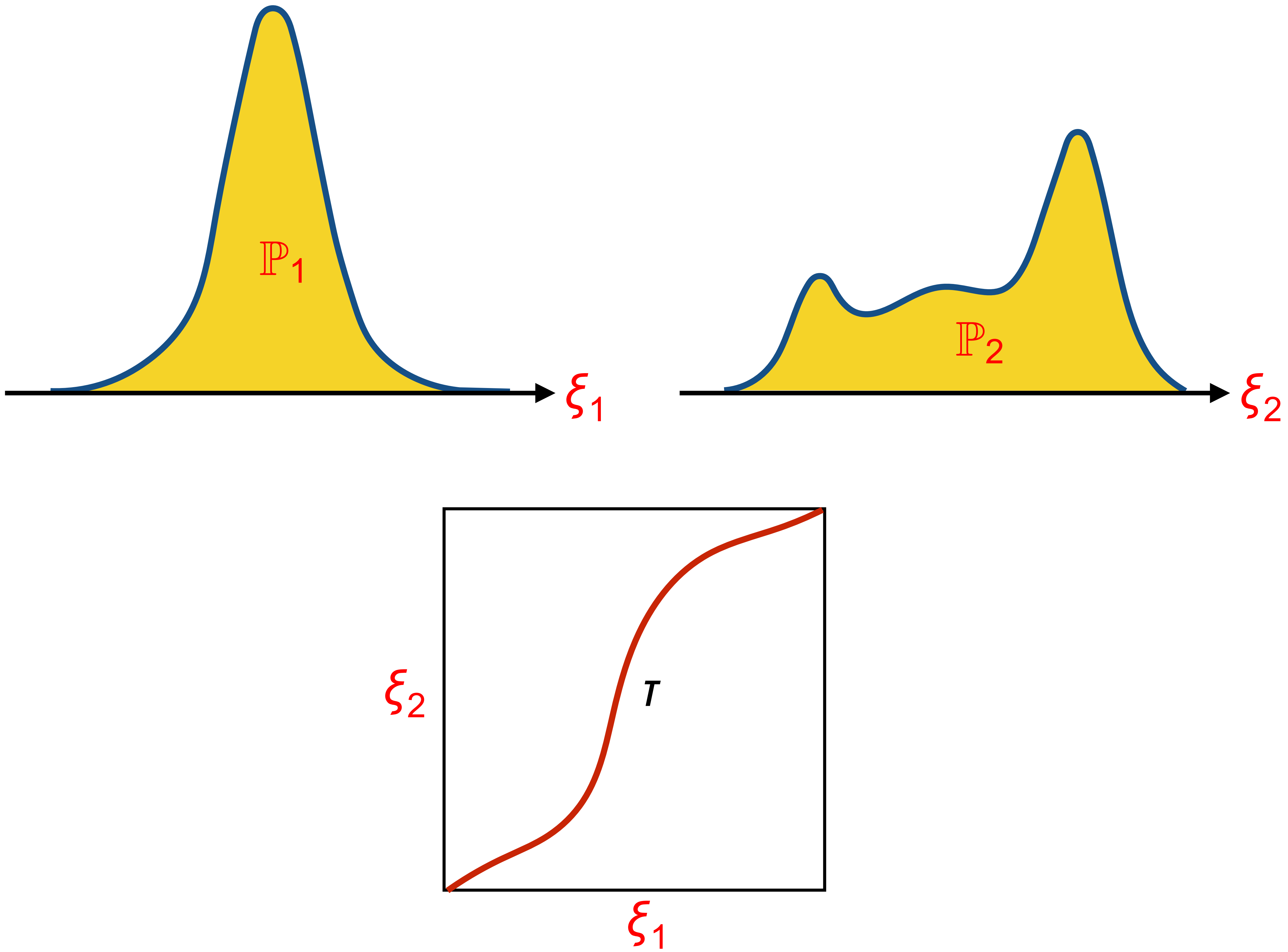

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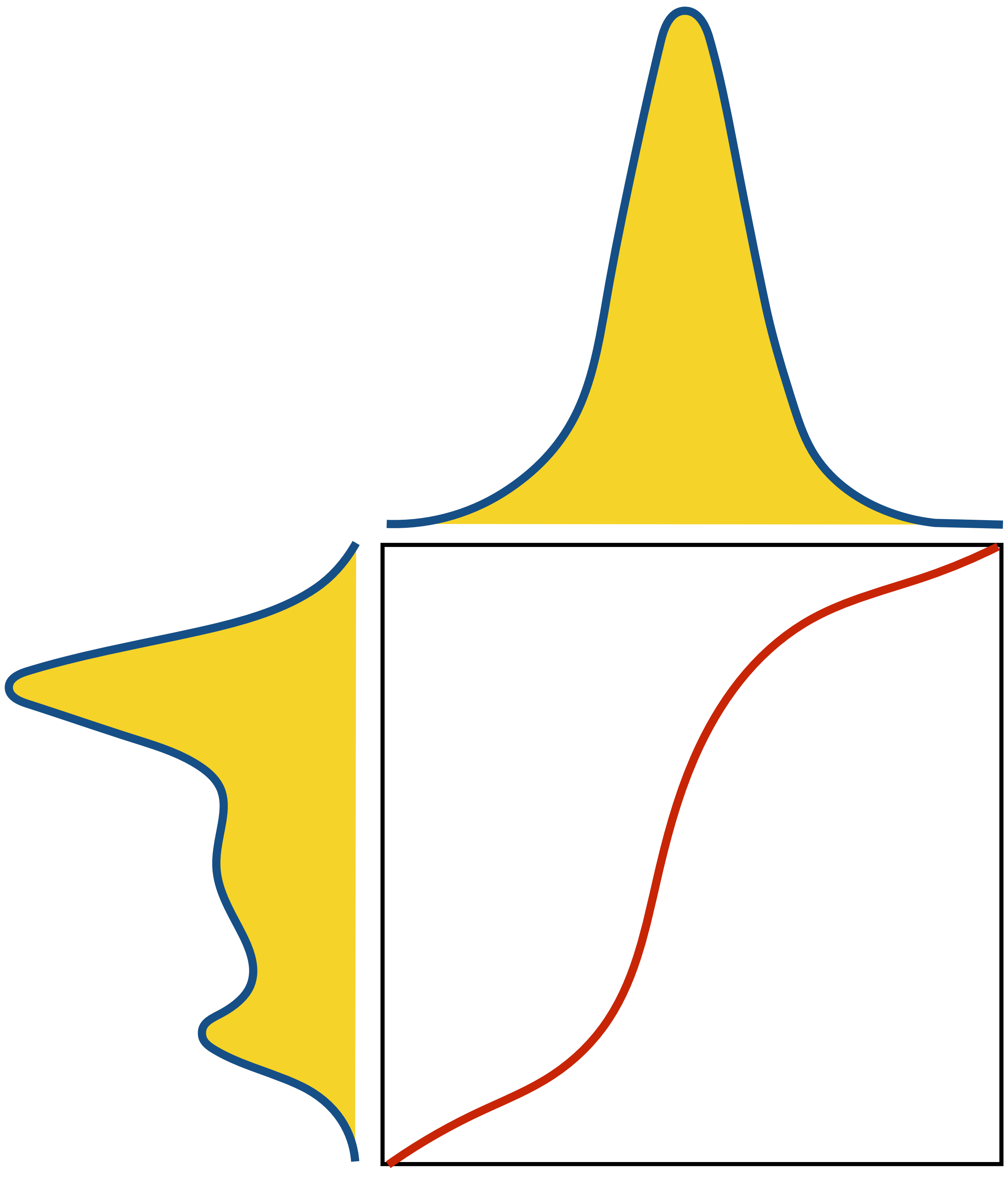
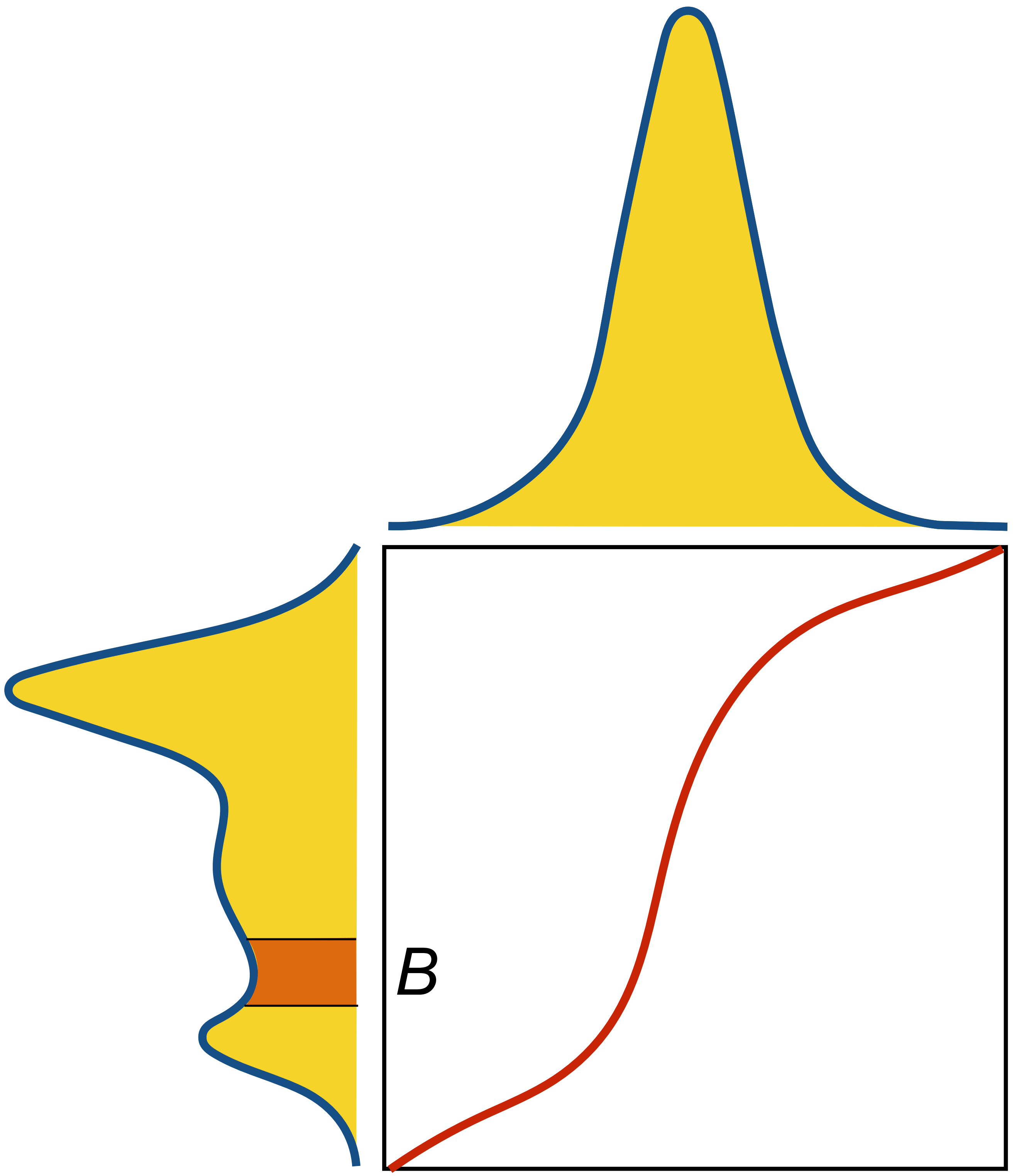


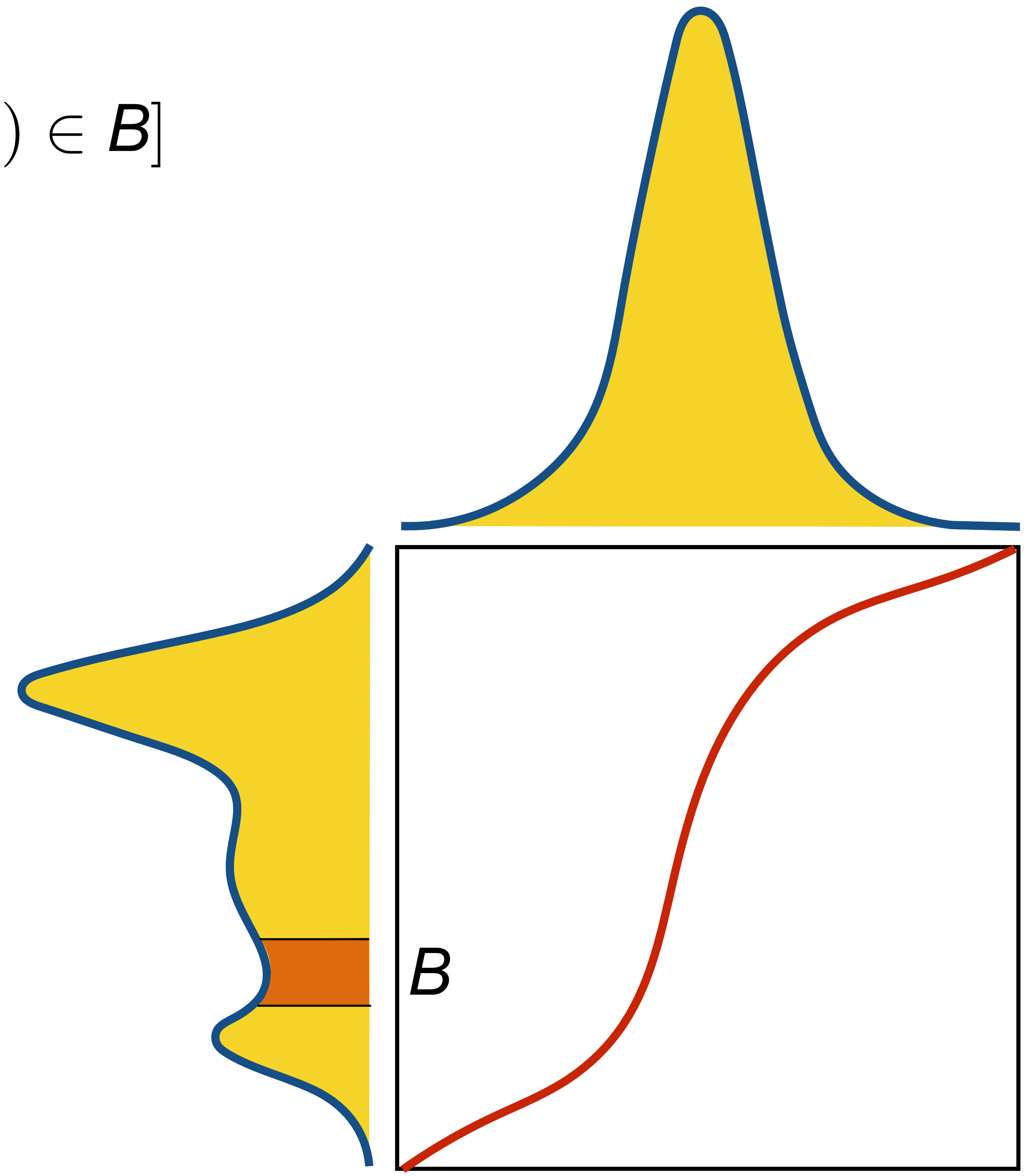
Pushforward Distribution

$\mathbb{P}_2[\xi_2 \in B]$



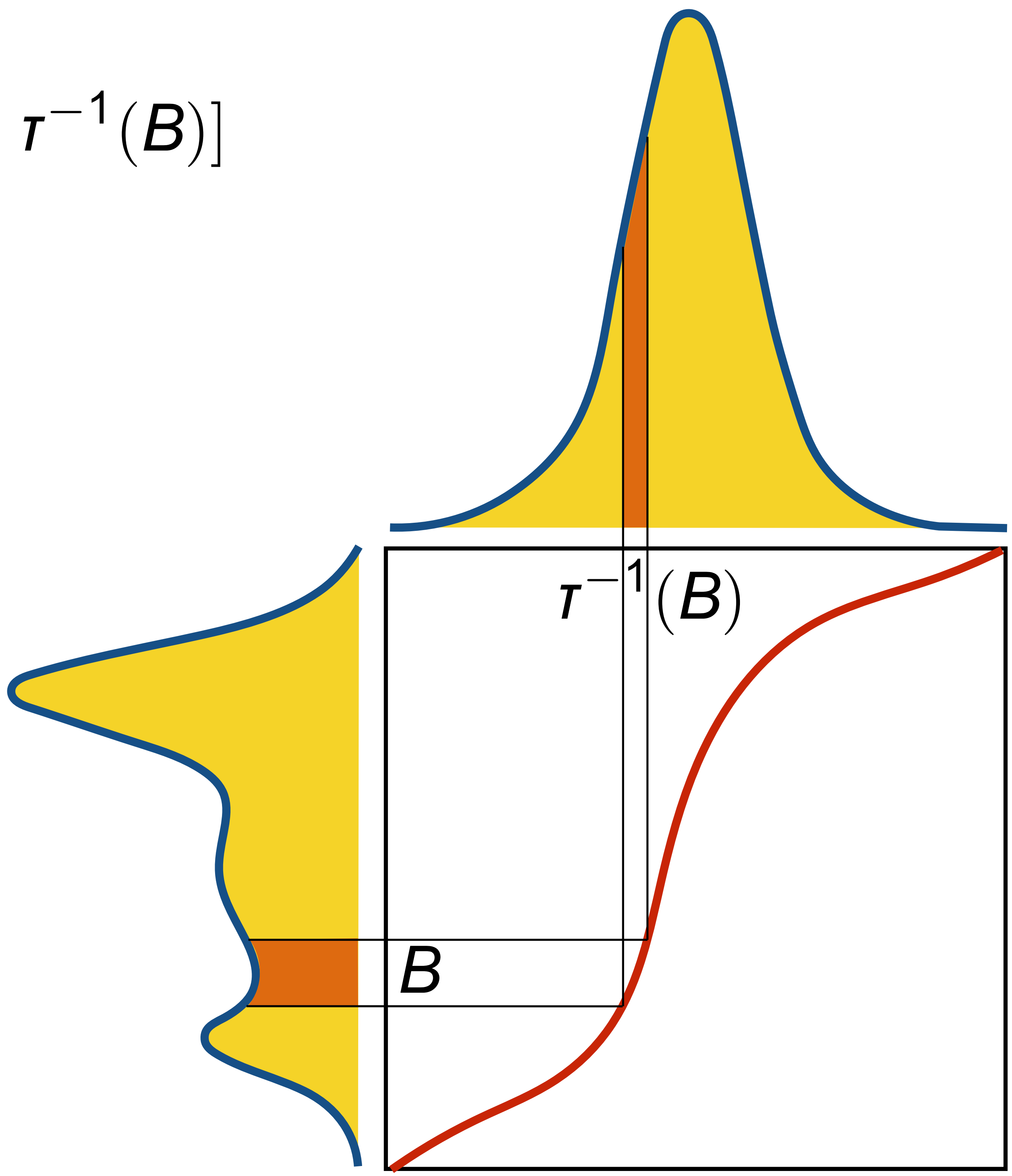
Pushforward Distribution

$$\mathbb{P}_2[\xi_2 \in B] = \mathbb{P}_1[\tau(\xi_1) \in B]$$



Pushforward Distribution

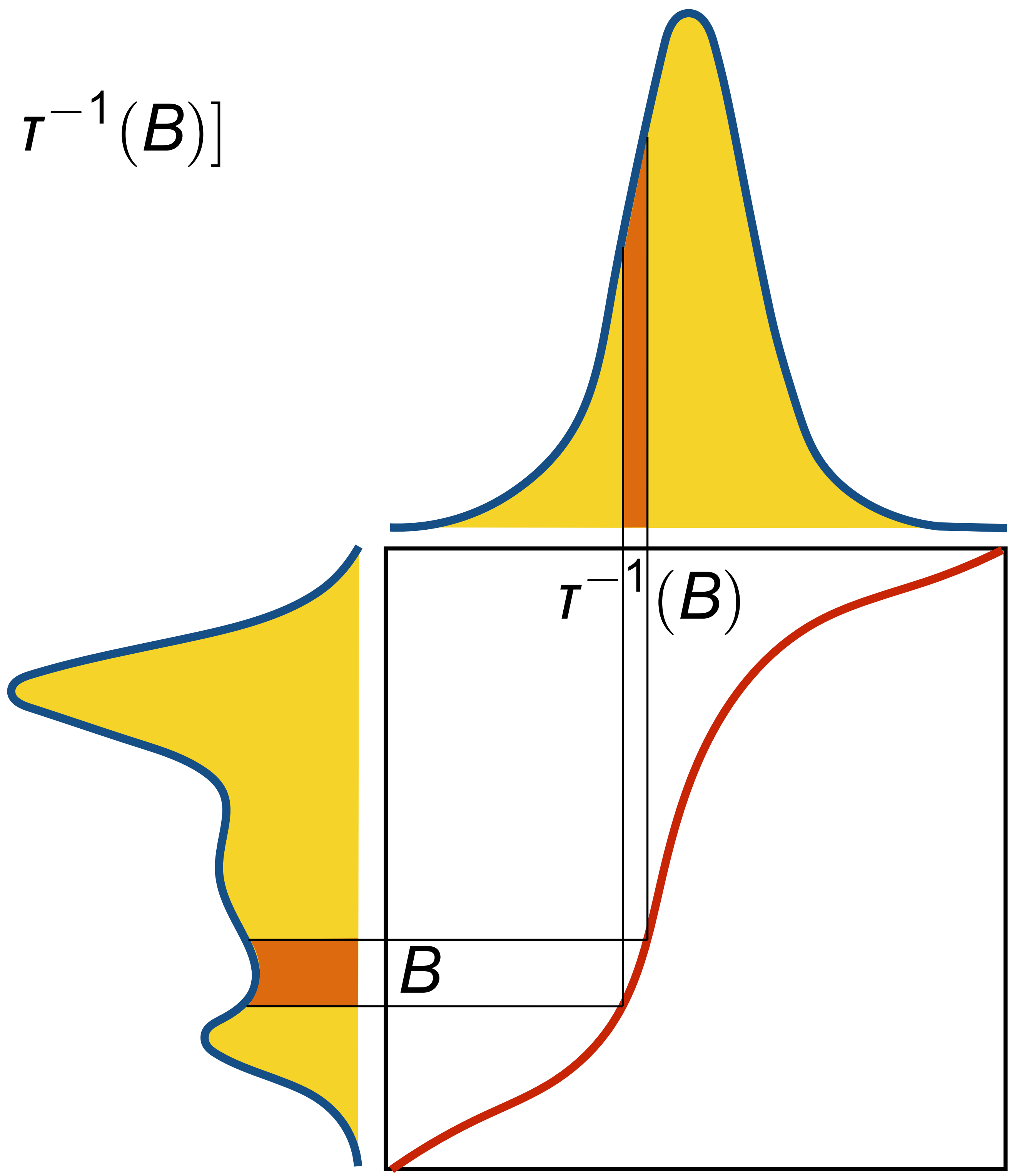
$$\mathbb{P}_2[\xi_2 \in B] = \mathbb{P}_1[\xi_1 \in \tau^{-1}(B)]$$



Pushforward Distribution

$$\mathbb{P}_2[\xi_2 \in B] = \mathbb{P}_1[\xi_1 \in \tau^{-1}(B)]$$

$$\implies \mathbb{P}_2 = \tau_\# \mathbb{P}_1$$

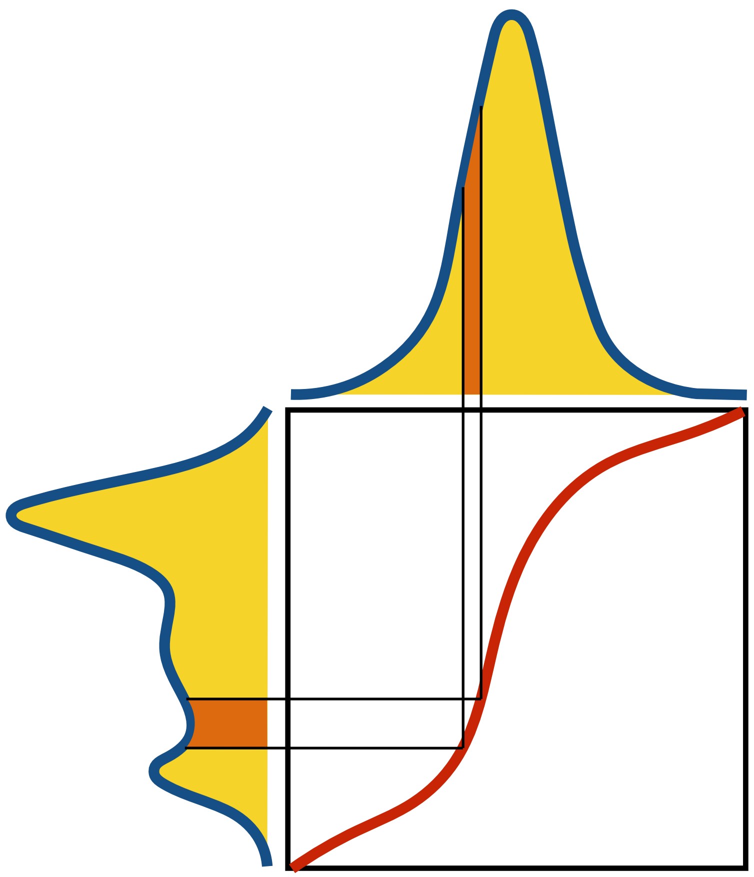


Monge's OT Problem

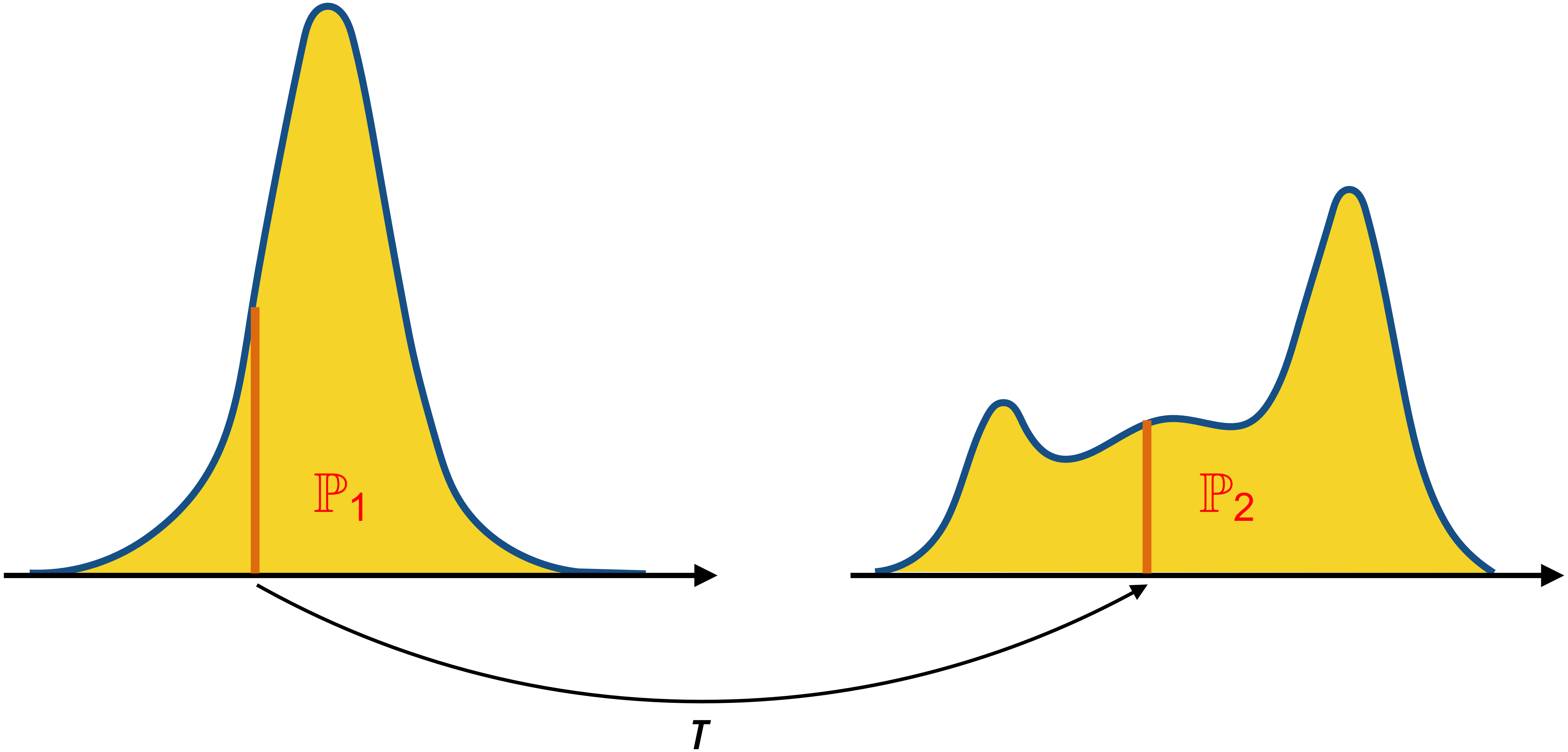
$$\inf_{T} \int c(\xi_1, T(\xi_1))\, \mathrm{d}\mathbb{P}_1(\xi_1)$$

$$\text{s.t.} \quad T_{\#}\mathbb{P}_1 = \mathbb{P}_2$$

Gaspard Monge
1784



Transportation Maps vs Transportation Plans



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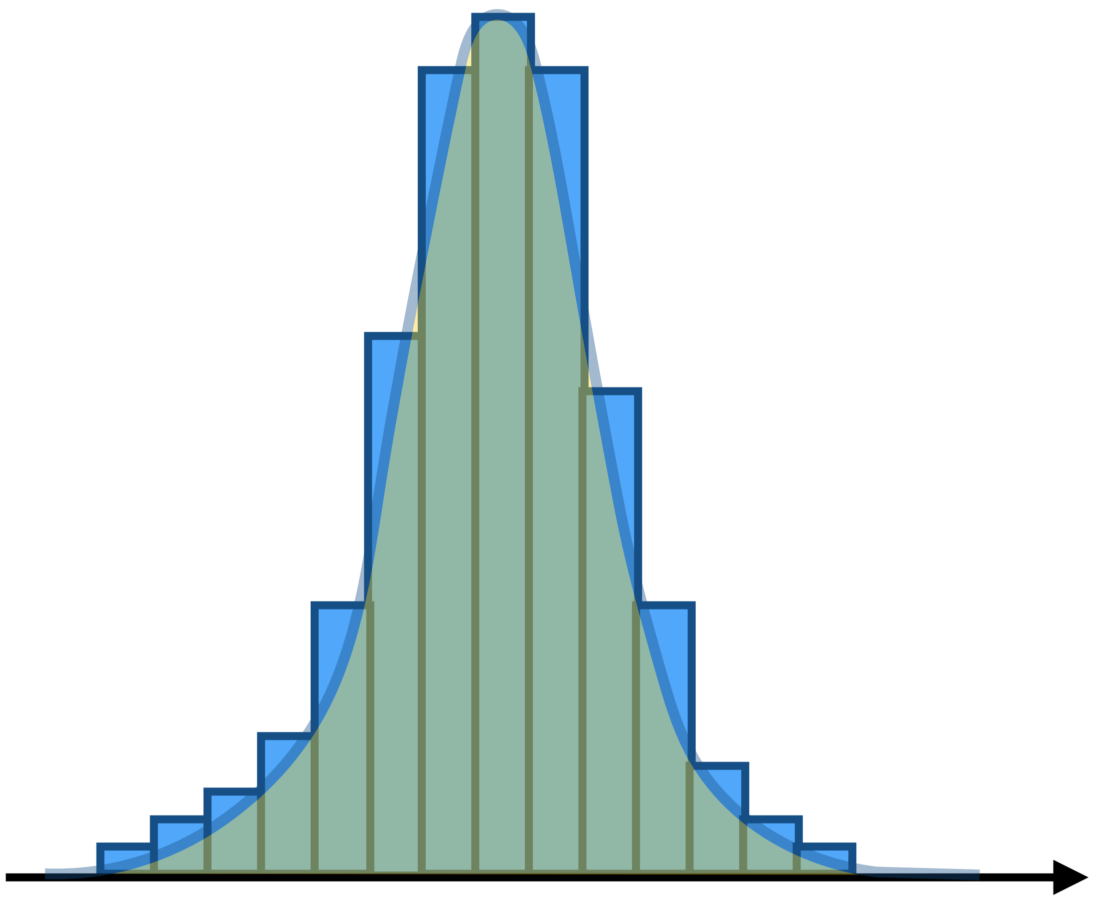

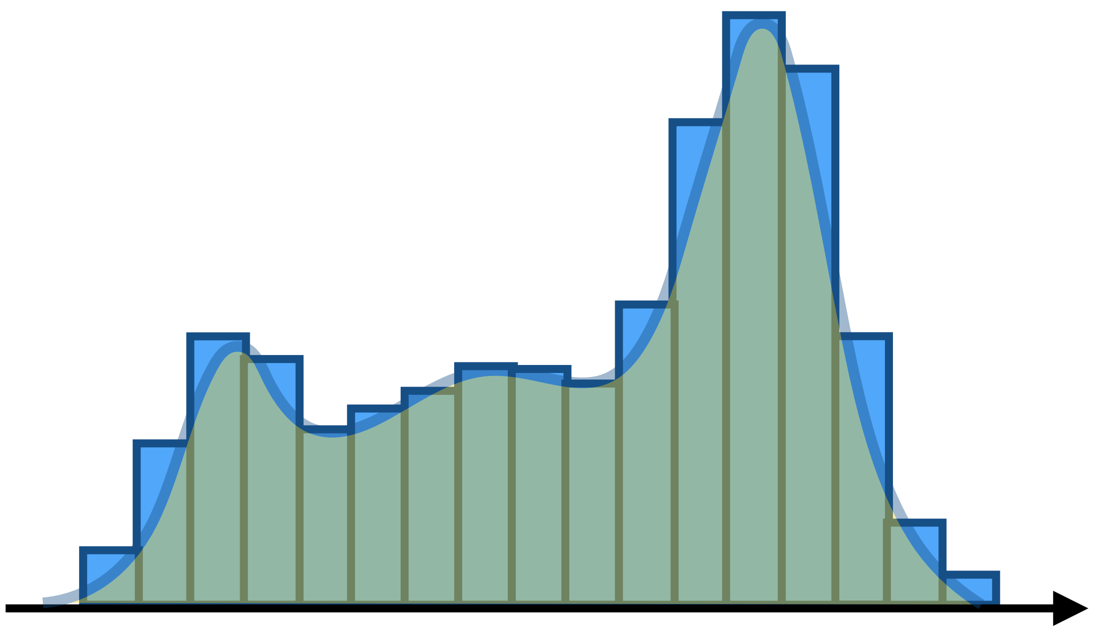

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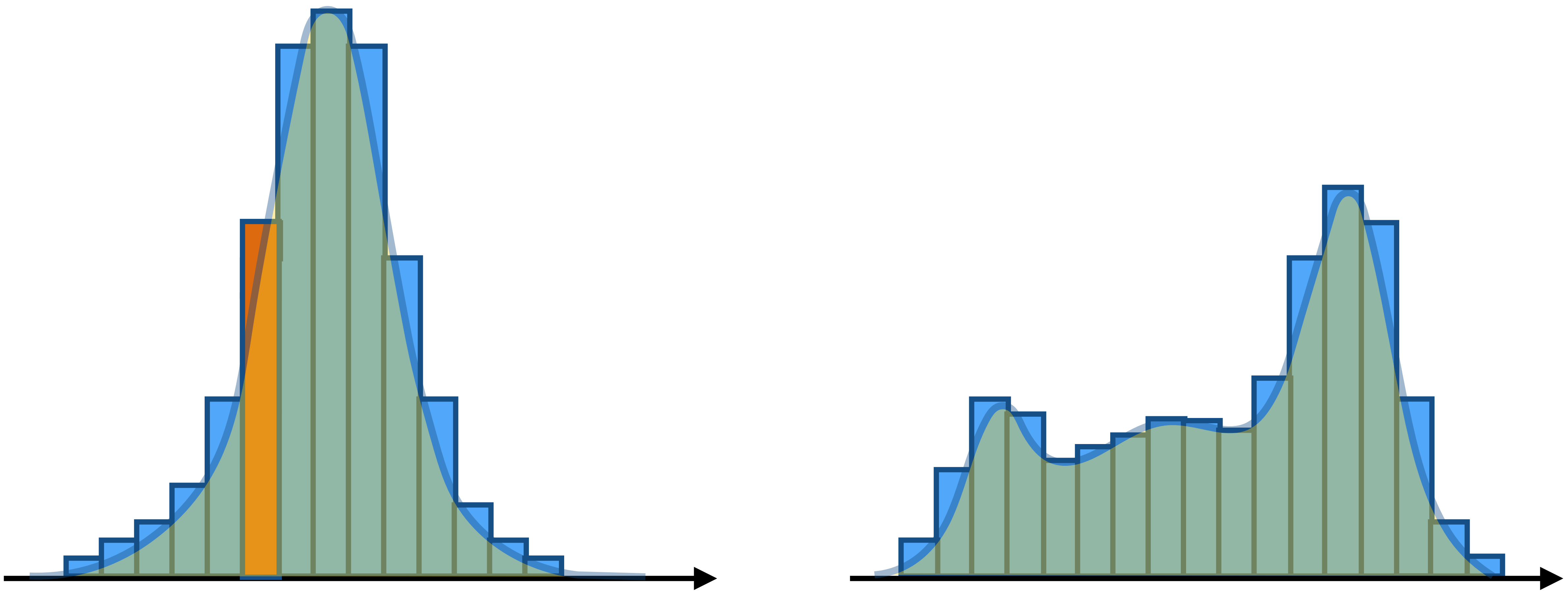

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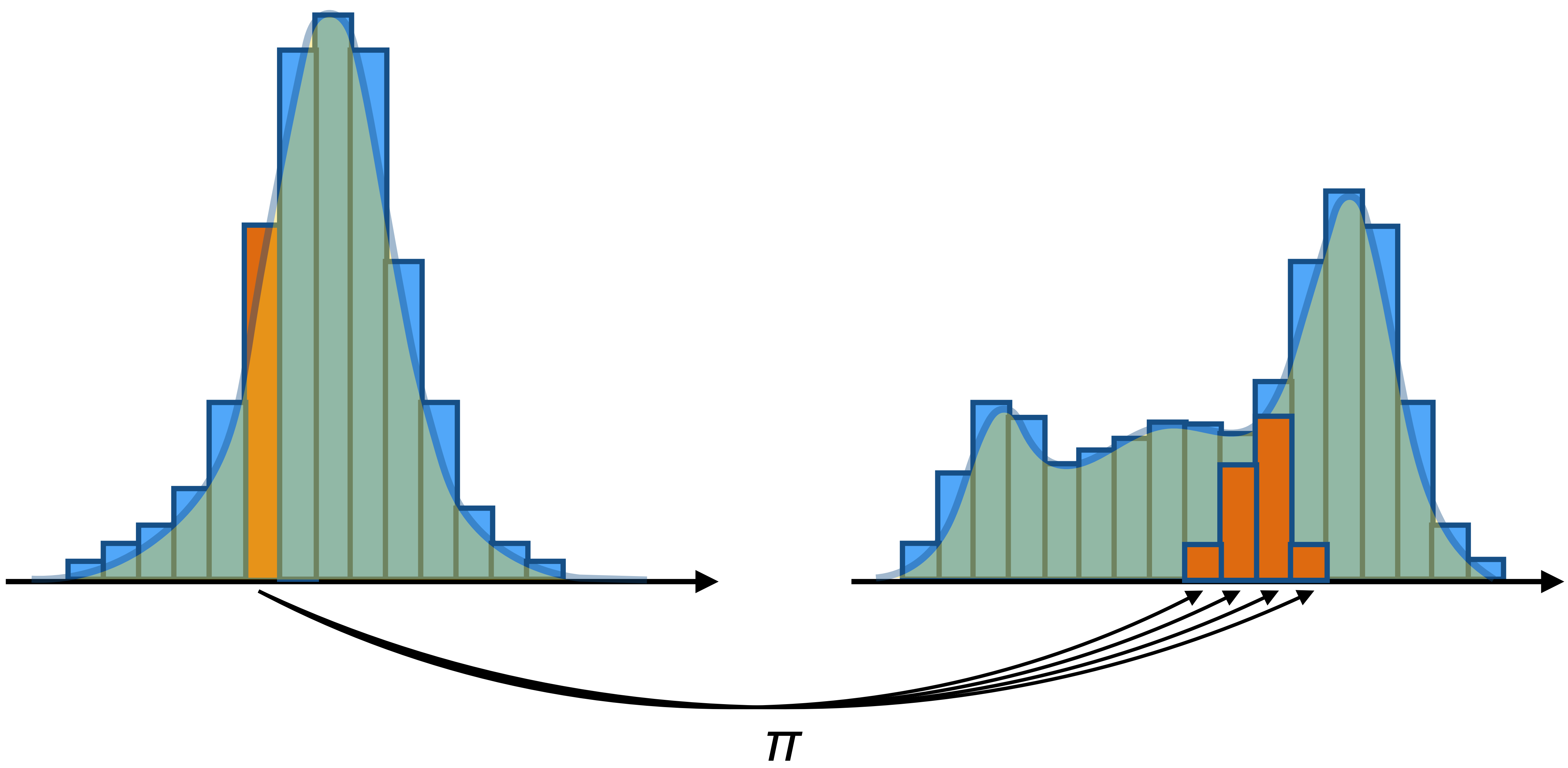

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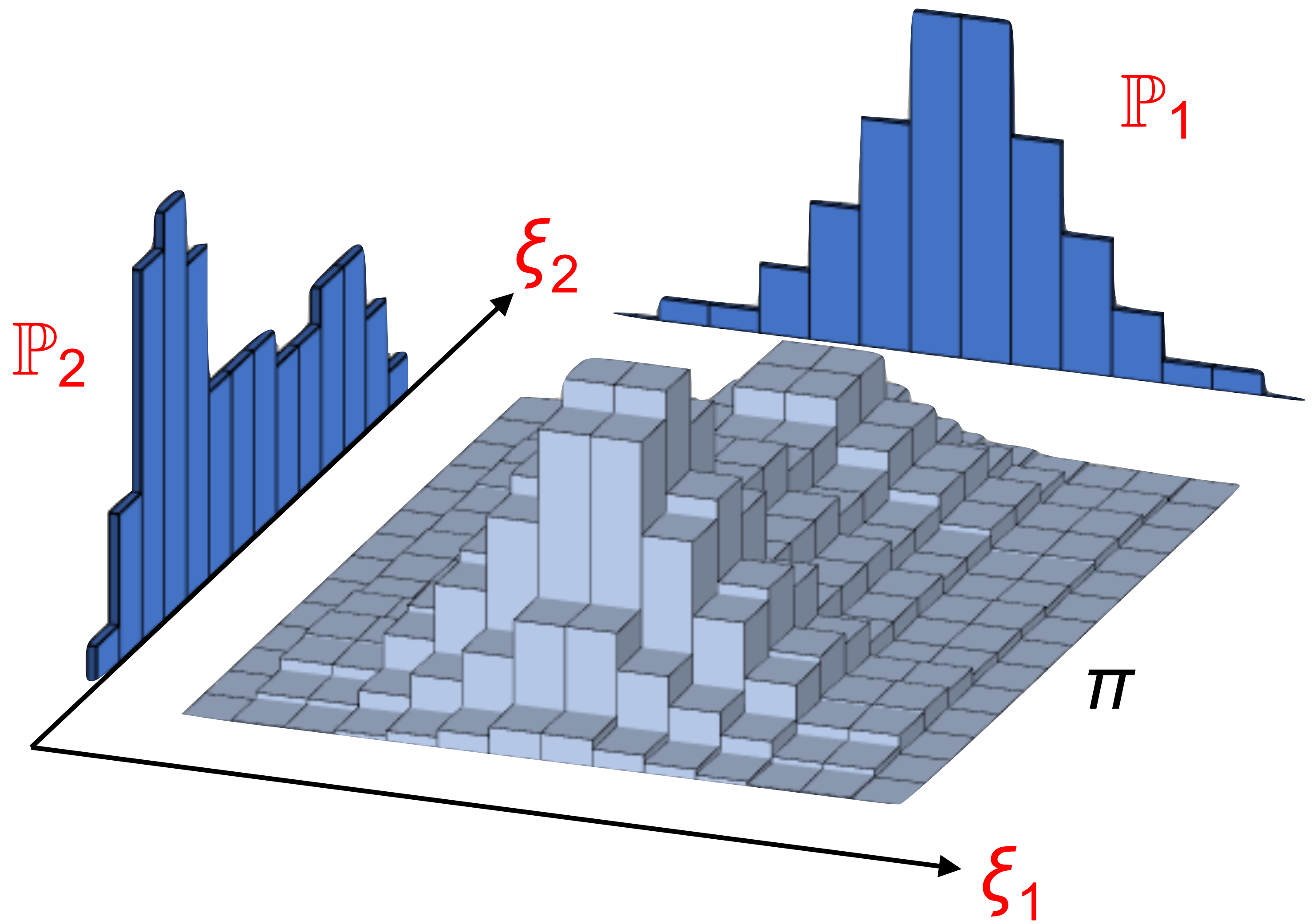

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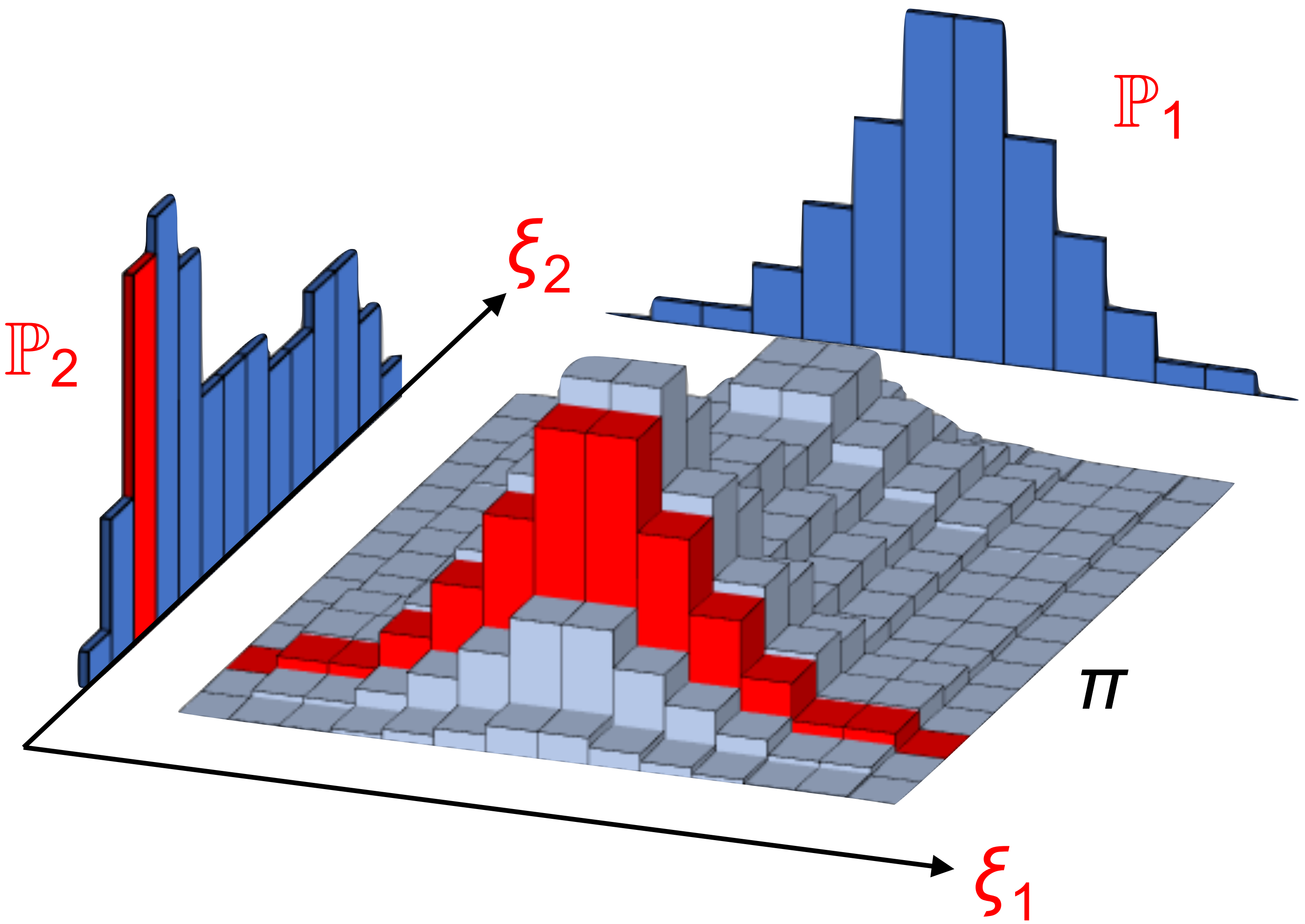

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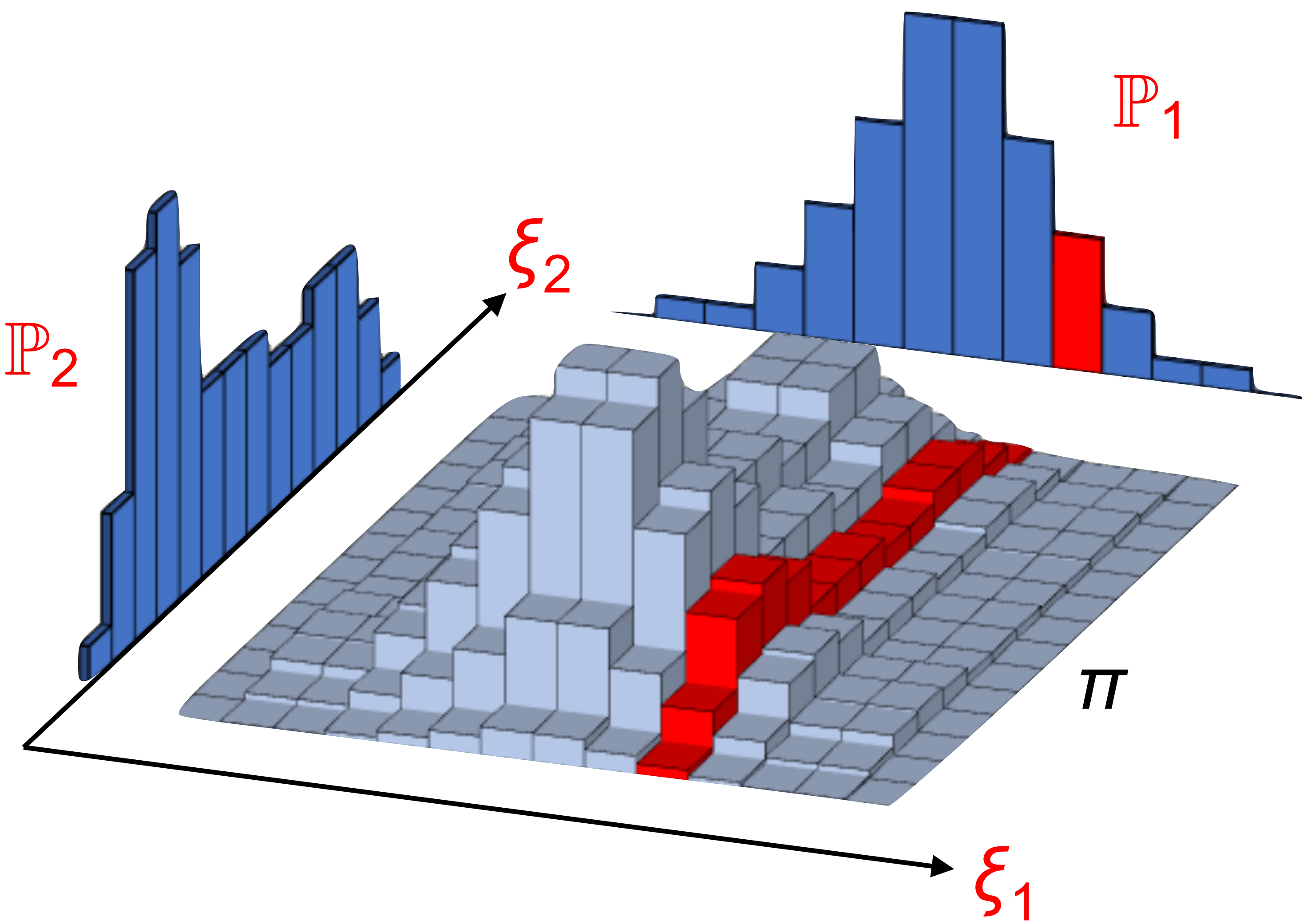
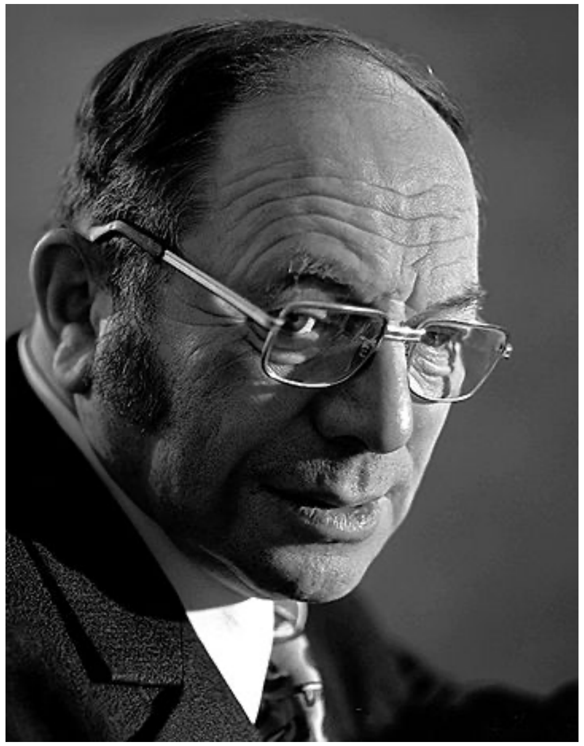
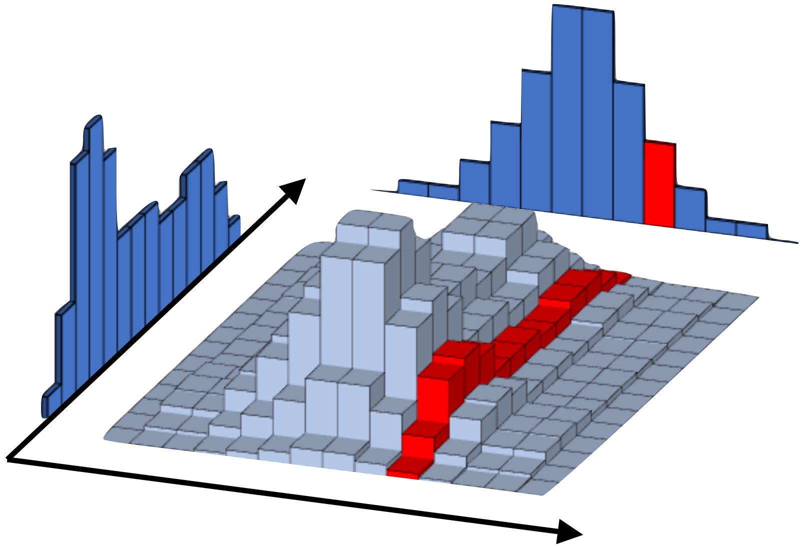


Kantorovich's OT Problem

$$\min_{\pi \in \Pi(\mathbb{P}_1, \mathbb{P}_2)} \int c(\xi_1, \xi_2)\, \mathrm{d}\pi(\xi_1, \xi_2)$$



Leonid Kantorovich
1942



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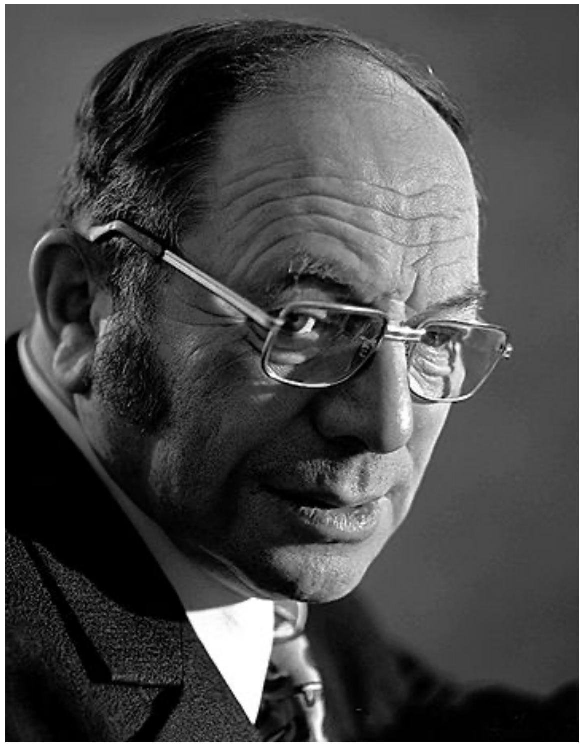
$$\min_{\pi \in \Pi(\mathbb{P}_1, \mathbb{P}_2)} \int c(\xi_1, \xi_2)\, \mathrm{d}\pi(\xi_1, \xi_2)$$

$\Longrightarrow$ linear program



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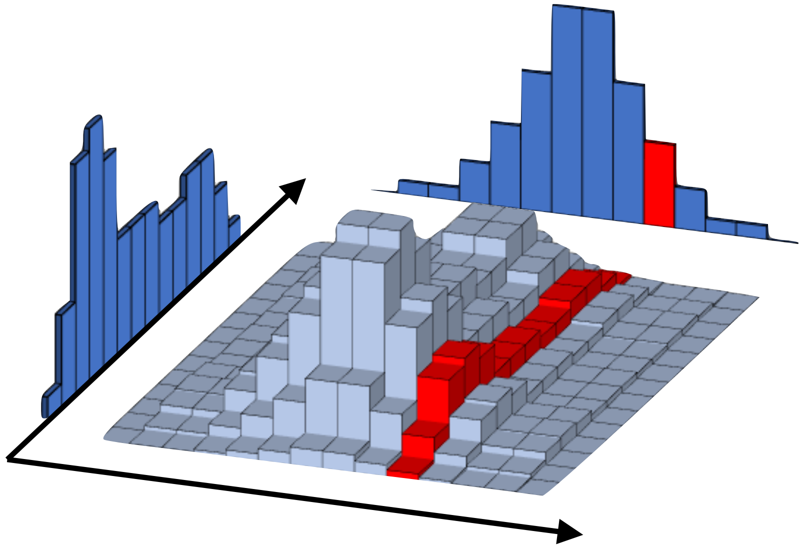

Wasserstein Distances

For $p \geq 1$, set

$$W_p(\mathbb{P}_1, \mathbb{P}_2) = \left(\min_{\pi \in \Pi(\mathbb{P}_1, \mathbb{P}_2)} \int d(\xi_1, \xi_2)^p \, \mathrm{d}\pi(\xi_1, \xi_2) \right)^{\frac{1}{p}}$$

Leonid Vaseršteĭn
1969

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metric on ξ-space

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metric on ξ-space
usually $d(\xi_1, \xi_2) = \|\xi_1 - \xi_2\|$

Leonid Vaseršteĭn
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metric on $\mathbb{P}$-space

Leonid Vaseršteĭn
1969

Computing Wasserstein Distances

Theorem: Computing $W_p(\mathbb{P}_1, \mathbb{P}_2)$ is #P-hard even if $\mathbb{P}_1 \sim \mathcal{U}[0,1]^d$ and $\mathbb{P}_2$ is a two-point distribution.[1)]

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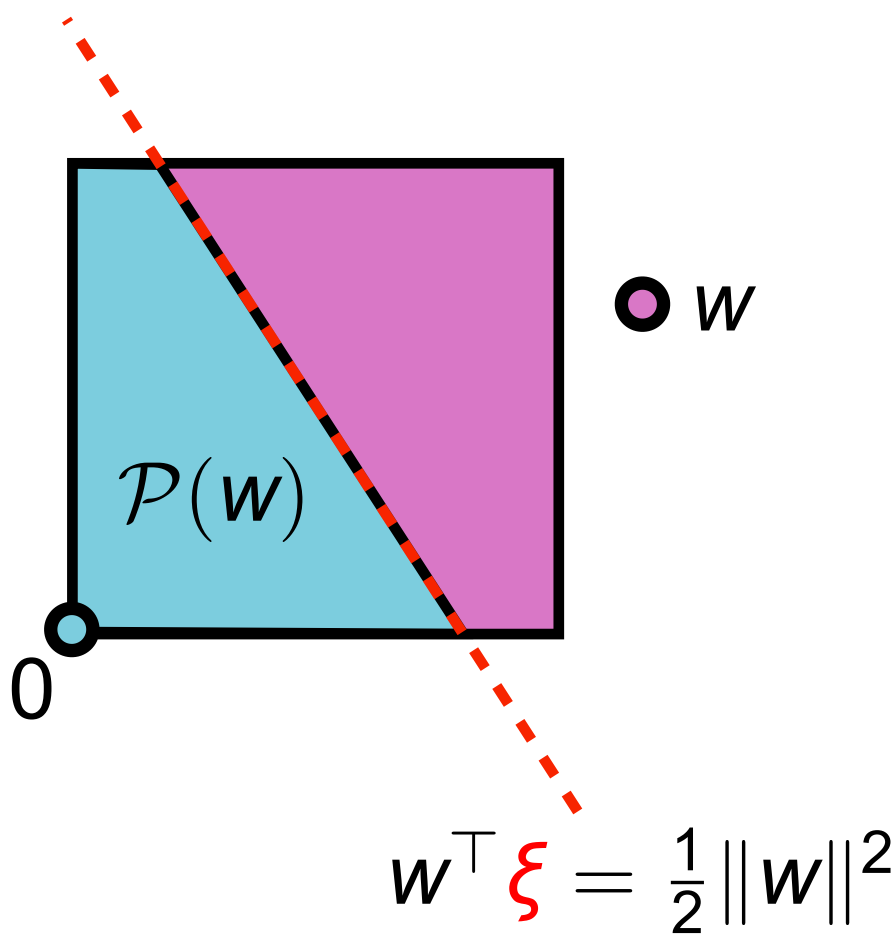
1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Computing Wasserstein Distances

Theorem: Computing $W_p(\mathbb{P}_1, \mathbb{P}_2)$ is #P-hard even if $\mathbb{P}_1 \sim \mathcal{U}[0,1]^d$ and $\mathbb{P}_2$ is a two-point distribution.[1)]

Proof: Computing the volume of the knapsack polytope $\mathcal{P}(w)$ is #P-hard.[2)]



$$\mathcal{P}(w) = \left\{ \xi \in [0,1]^d : w^\top \xi \leq \tfrac{1}{2}\|w\|^2 \right\}$$

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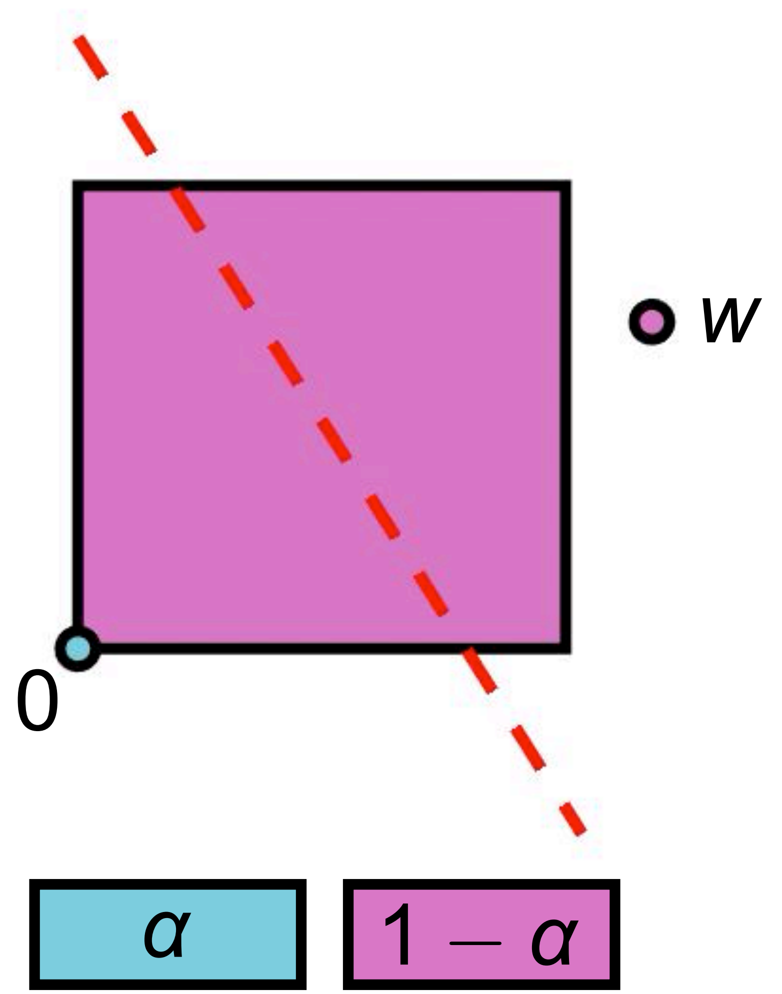
1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

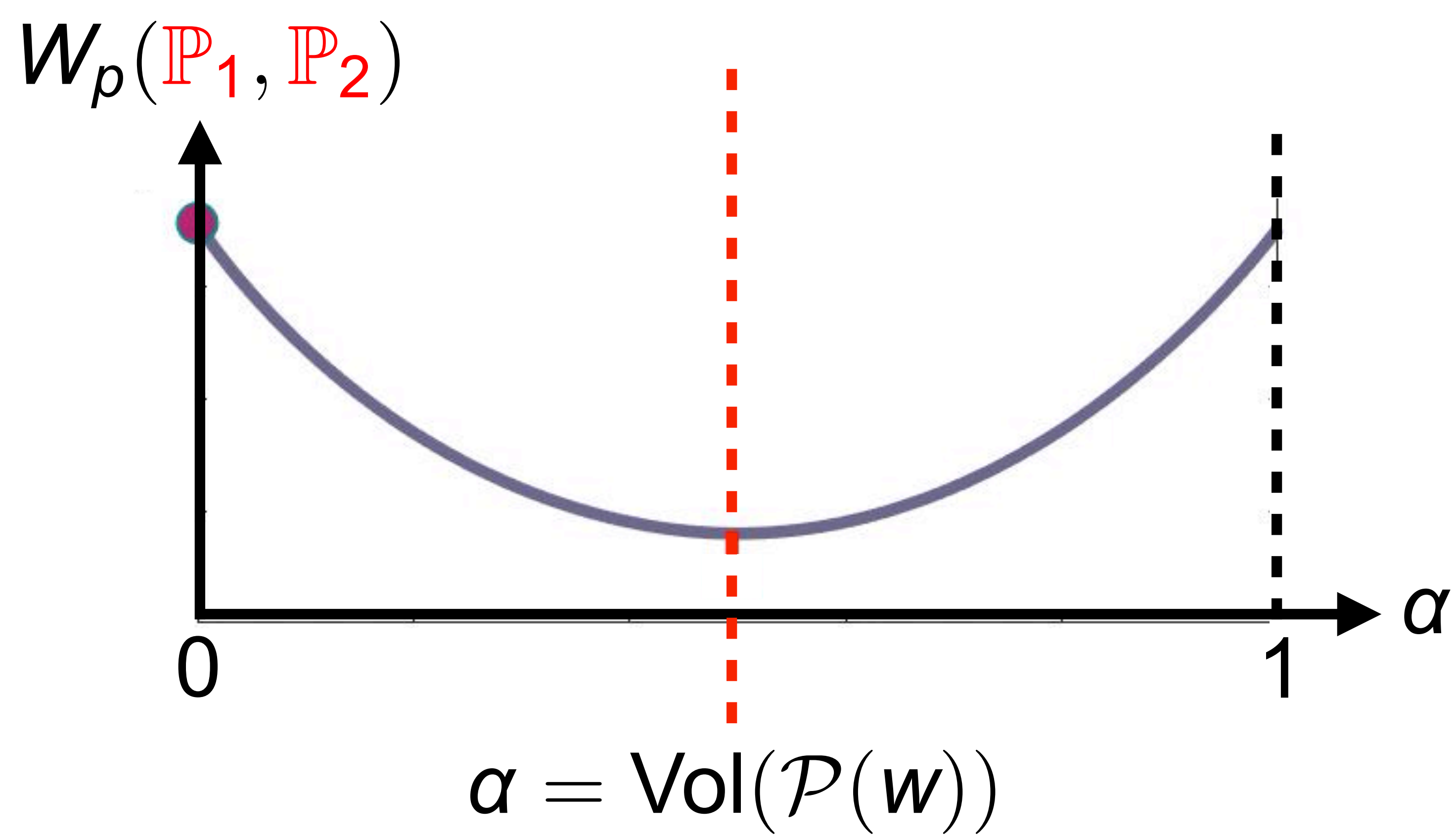
2) Dyer & Frieze, *SIAM J. Comp.*, 1988.

Computing Wasserstein Distances

Theorem: Computing $W_p(\mathbb{P}_1, \mathbb{P}_2)$ is #P-hard even if $\mathbb{P}_1 \sim \mathcal{U}[0,1]^d$ and $\mathbb{P}_2$ is a two-point distribution.[1)]

Proof: $\mathbb{P}_2 = \alpha \cdot \delta_0 + (1-\alpha) \cdot \delta_w$





1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Brenier's Theorem

Theorem (Brenier 1987): If $c(\xi_1, \xi_2) = \|\xi_1 - \xi_2\|_2^2$, then

- $T^\star$ solves the Monge problem iff
- $T^\star$ is feasible and $\exists \varphi$ convex such that $T^\star = \nabla \varphi$.

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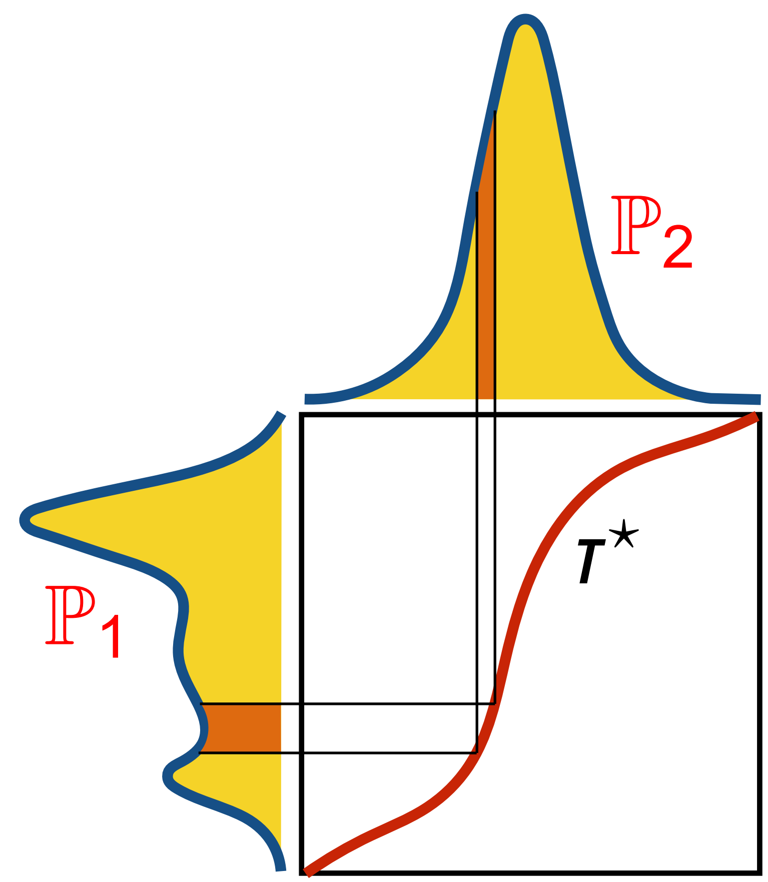
$\mathbb{P}_2 = T^\star_\# \mathbb{P}_1$



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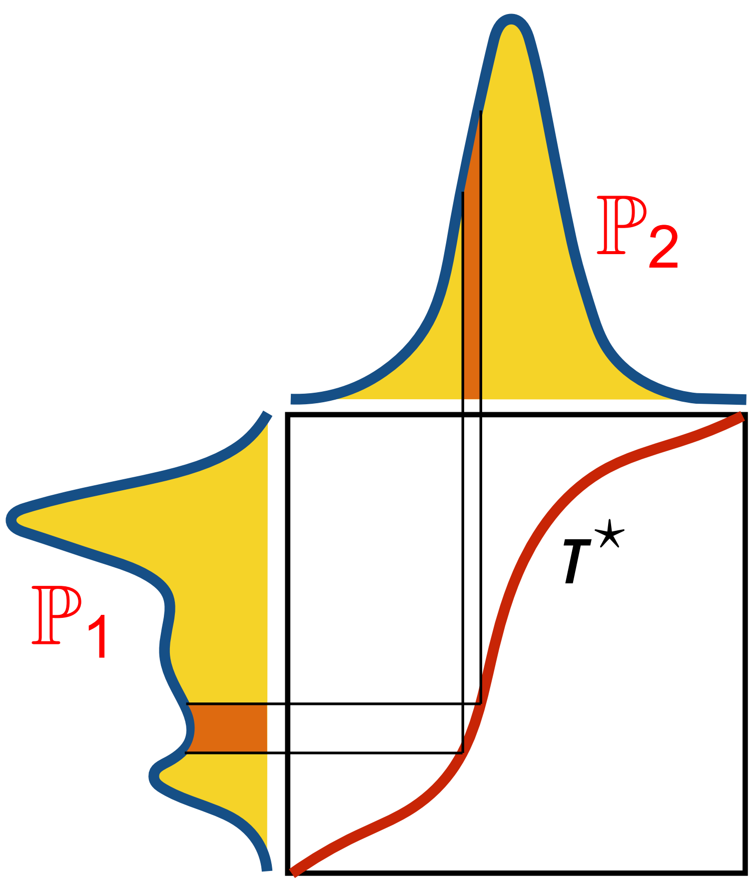
optimality



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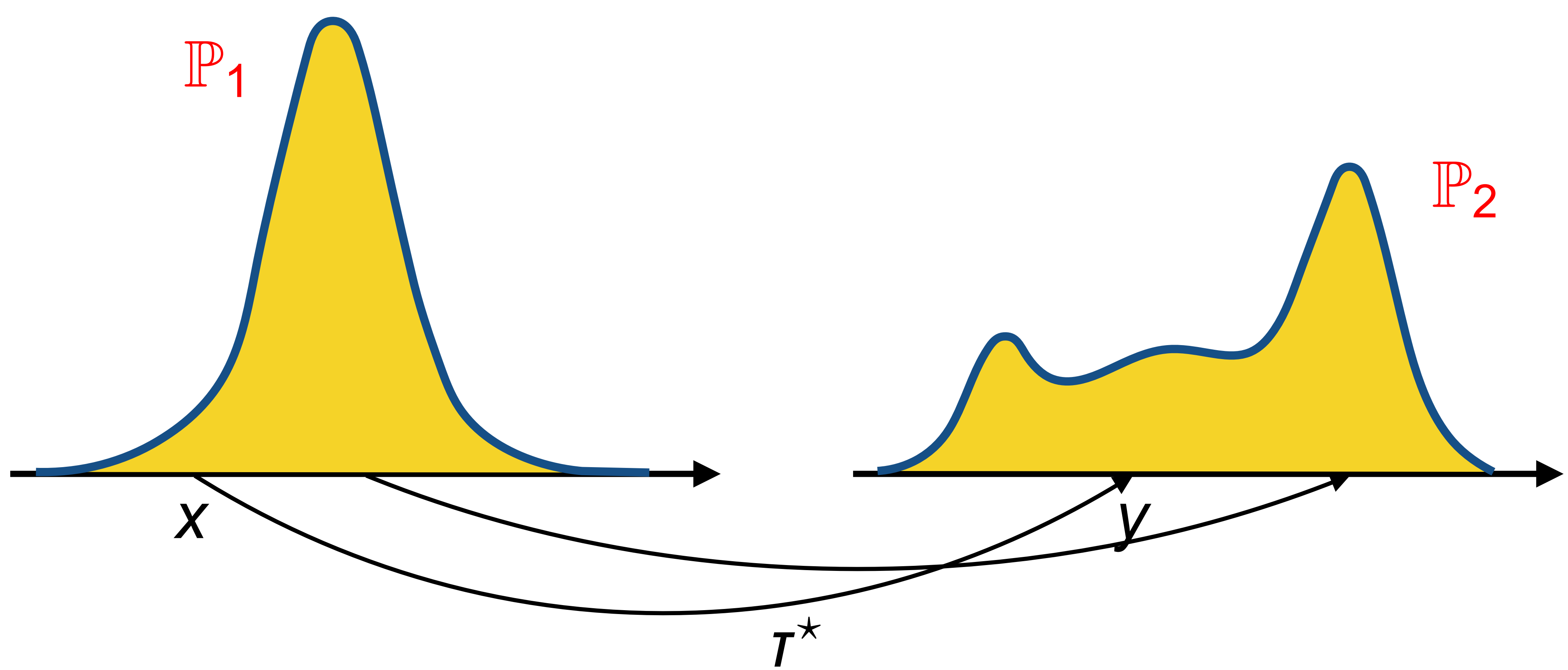
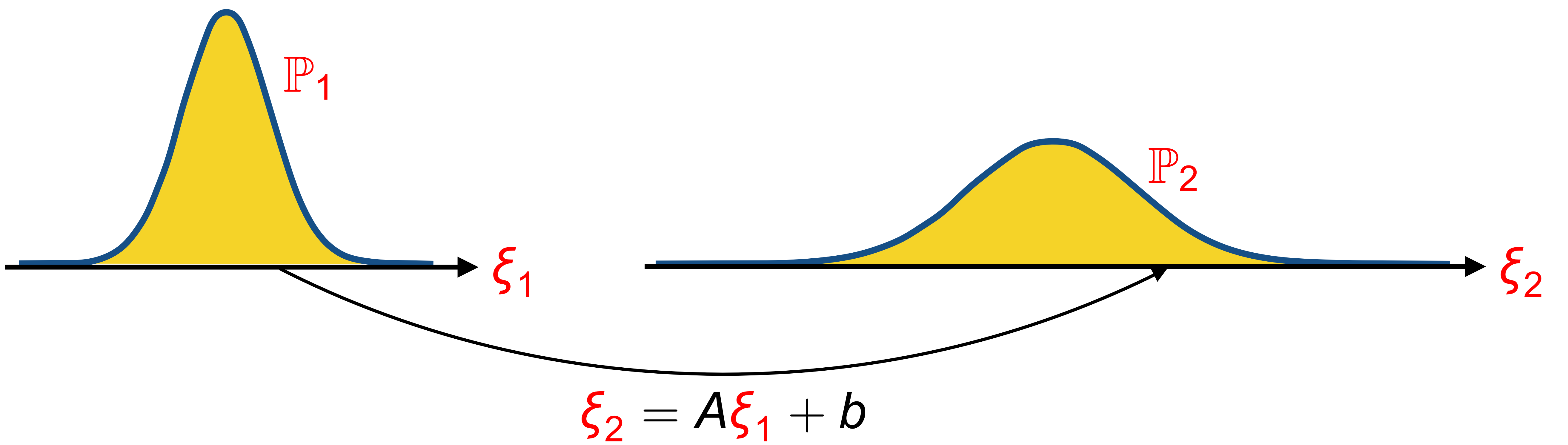


2-Wasserstein Distance in Closed Form



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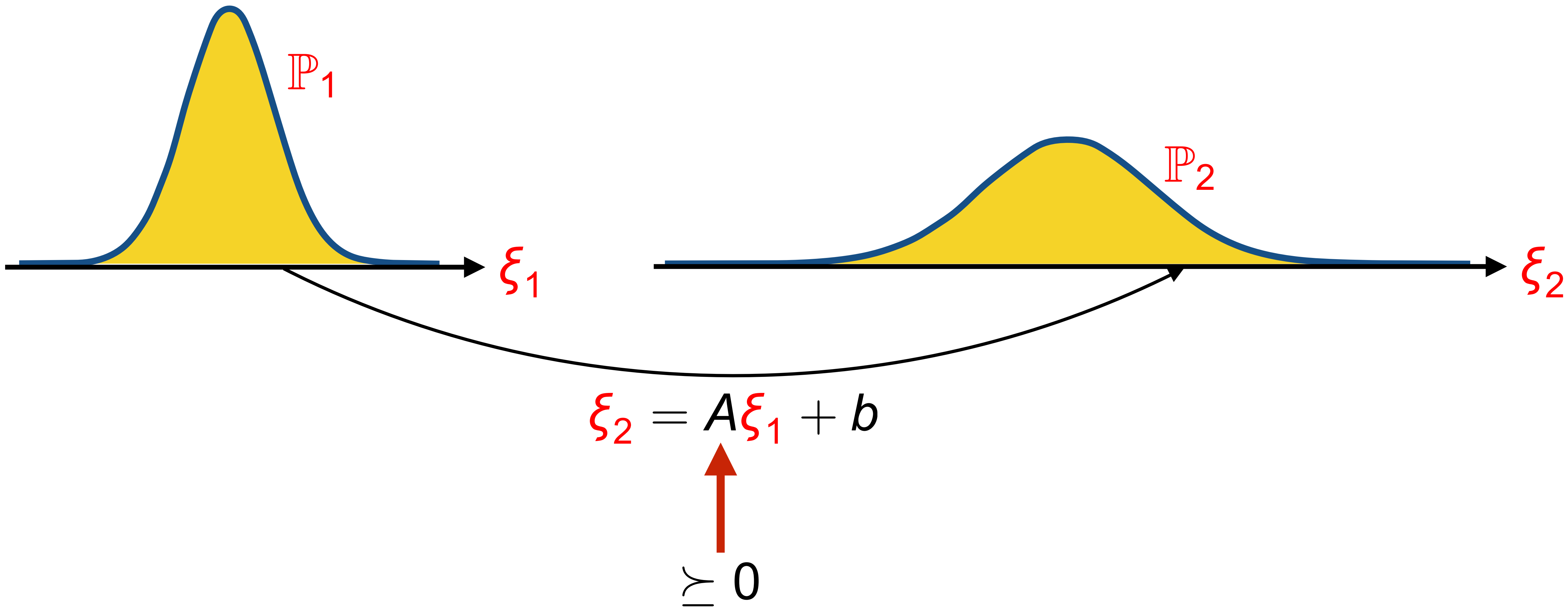

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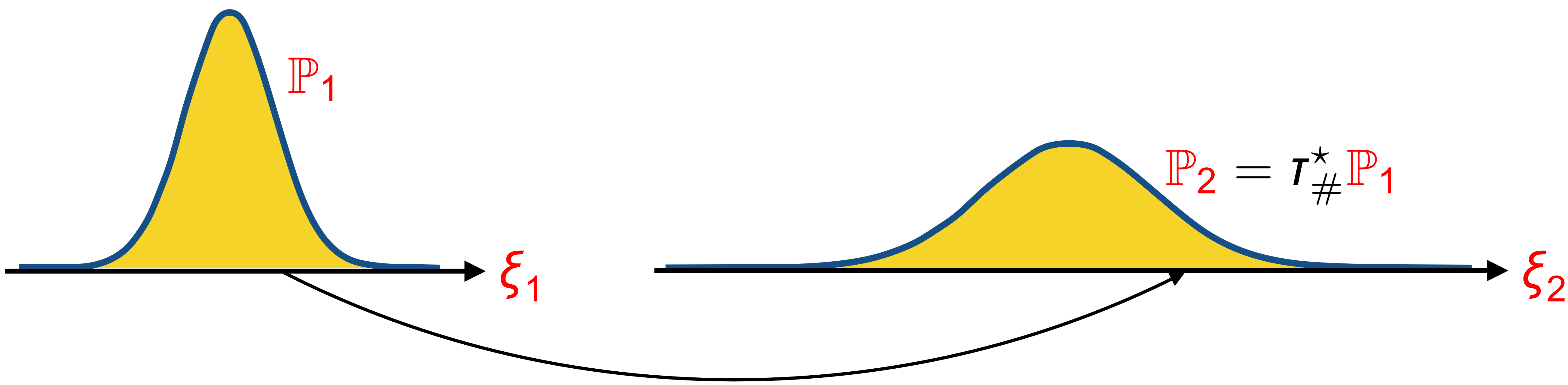
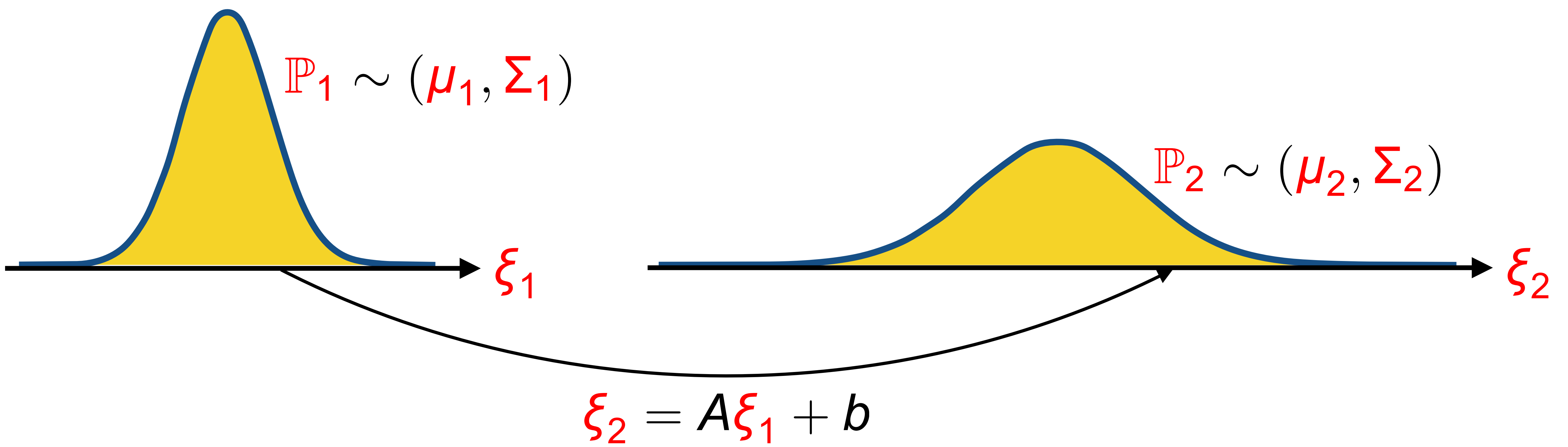


$$\xi_2 = \underbrace{A\xi_1 + b}_{= \tau^\star(\xi_1) \text{ by Brenier!}}$$

2-Wasserstein Distance in Closed Form



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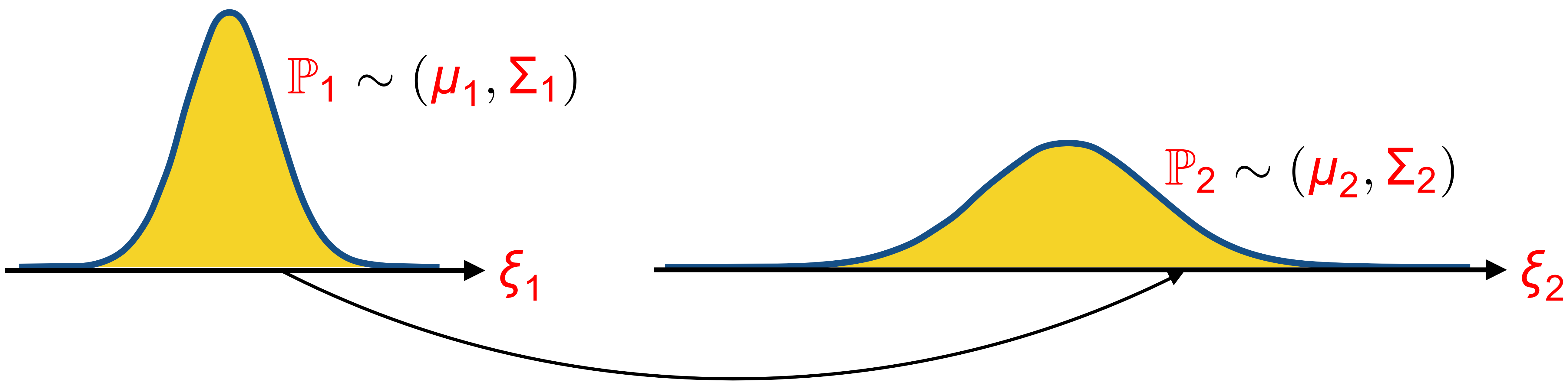
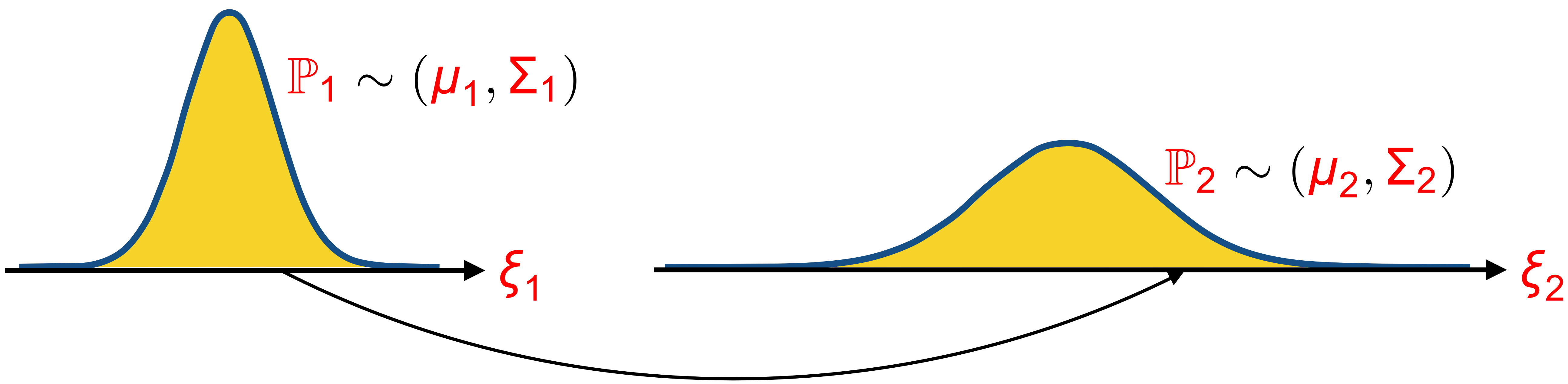


$$\xi_2 = A\xi_1 + b$$

$$\Sigma_1^{-\frac{1}{2}} \left(\Sigma_1^{\frac{1}{2}} \Sigma_2 \Sigma_1^{\frac{1}{2}} \right)^{\frac{1}{2}} \Sigma_1^{-\frac{1}{2}}$$

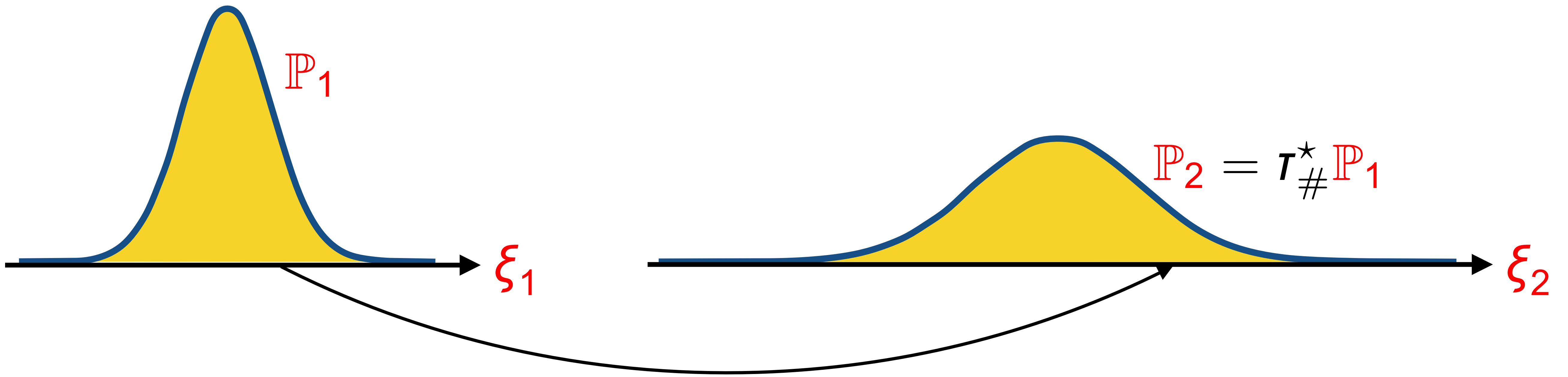
2-Wasserstein Distance in Closed Form



$$\xi_2 = A\xi_1 + b$$

$$\mu_2 - A\mu_1$$

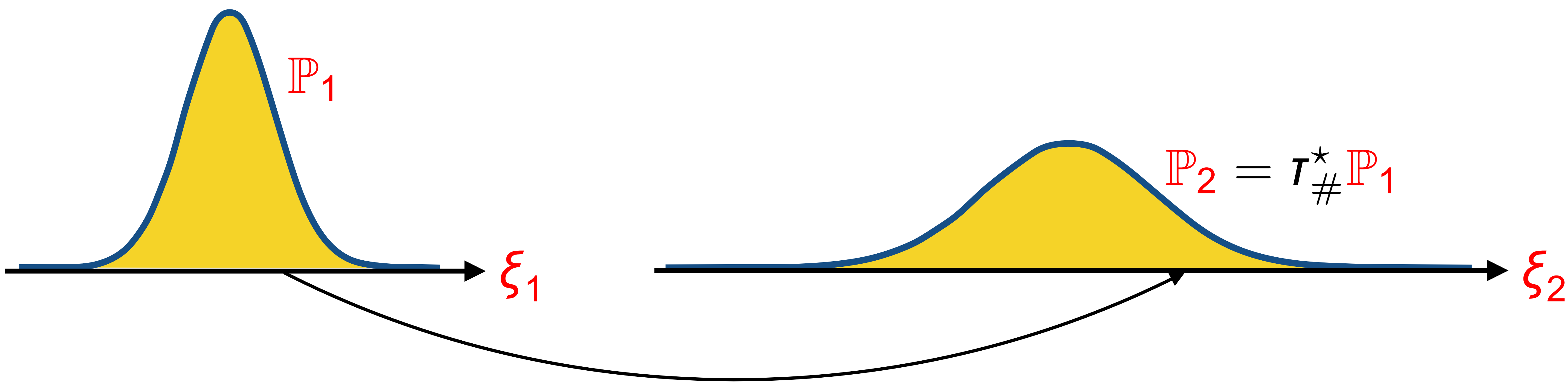
2-Wasserstein Distance in Closed Form



$$\xi_2 = \underbrace{A\xi_1 + b}_{= \tau^\star(\xi_1)}$$

$$\Longrightarrow \quad W_2^2(\mathbb{P}_1, \mathbb{P}_2) = \int \|\xi_1 - \tau^\star(\xi_1)\|_2^2 \, \mathrm{d}\mathbb{P}_1(\xi_1)$$

2-Wasserstein Distance in Closed Form



$$\xi_2 = \underbrace{A\xi_1 + b}_{= T^\star(\xi_1)}$$

$$\Longrightarrow \quad W_2^2(\mathbb{P}_1, \mathbb{P}_2) = \|\mu_1 - \mu_2\|_2^2 + \mathrm{tr}\left[\Sigma_1 + \Sigma_2 - 2\left(\Sigma_1^{\frac{1}{2}} \Sigma_2 \Sigma_1^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

2-Wasserstein Distance in Closed Form

Corollary 1: If $\mathbb{P}_2$ is a psd affine pushforward of $\mathbb{P}_1$, then

$$W_2^2(\mathbb{P}_1, \mathbb{P}_2) = \|\mu_1 - \mu_2\|_2^2 + \operatorname{tr}\left[\Sigma_1 + \Sigma_2 - 2\left(\Sigma_1^{\frac{1}{2}}\Sigma_2\Sigma_1^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

2-Wasserstein Distance in Closed Form

Corollary 2: If $\mathbb{P}_2$ and $\mathbb{P}_1$ are arbitrary distributions, then[1)]

$$W_2^2(\mathbb{P}_1, \mathbb{P}_2) \geq \|\mu_1 - \mu_2\|_2^2 + \mathrm{tr}\left[\Sigma_1 + \Sigma_2 - 2\left(\Sigma_1^{\frac{1}{2}}\Sigma_2\Sigma_1^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$$

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1) Gelbrich, *Mathematische Nachrichten*, 1990.

Distributionally Robust Optimization (DRO)

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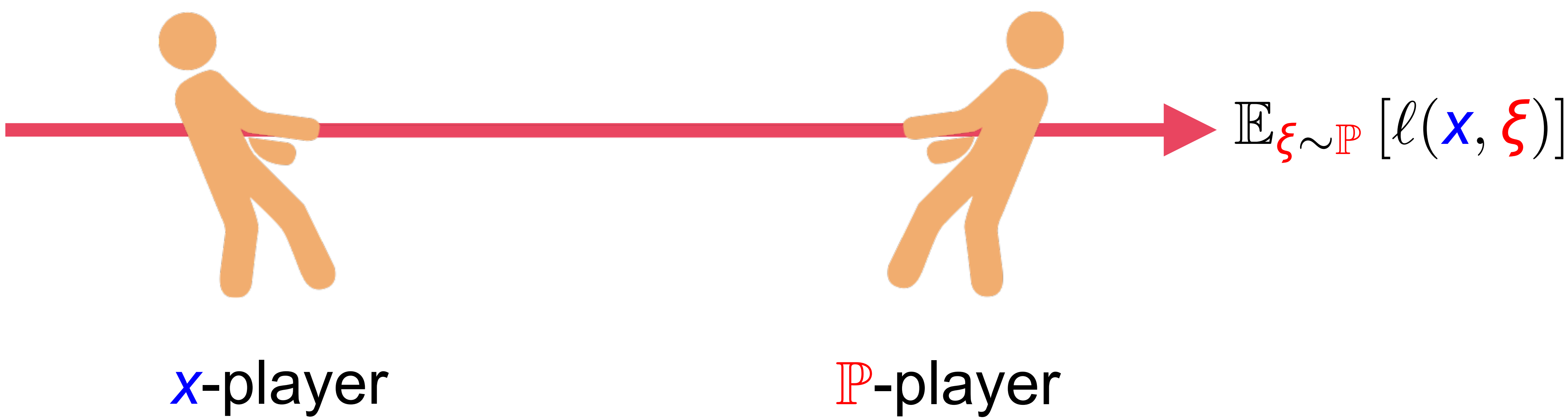
Zero-sum game against "nature":

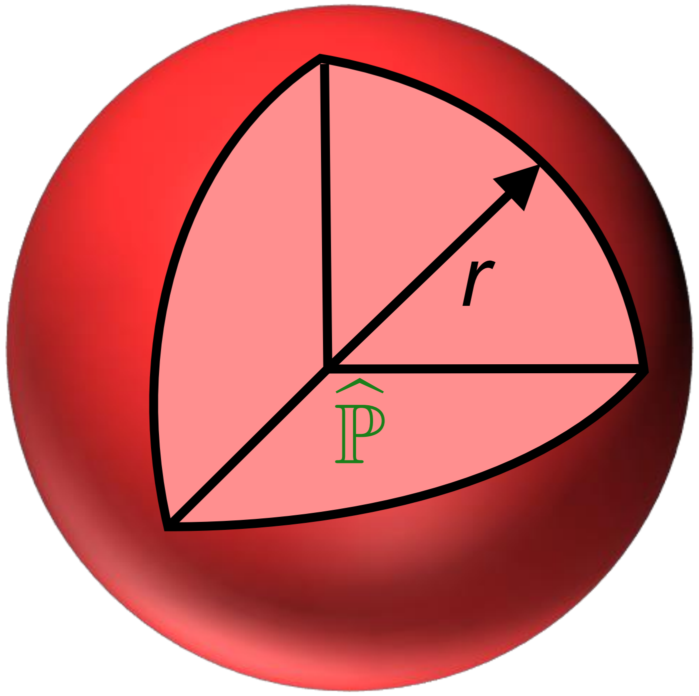
$$\inf_{x \in \mathcal{X}} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)]$$



Wasserstein DRO

Zero-sum game against "nature":

$$\inf_{x \in \mathcal{X}} \sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} \left[\ell(x, \xi) \right]$$



1) Pflug & Wozabal, *Quant. Finance*, 2007; Pflug & Pichler, *Springer*, 2014.

Wasserstein DRO

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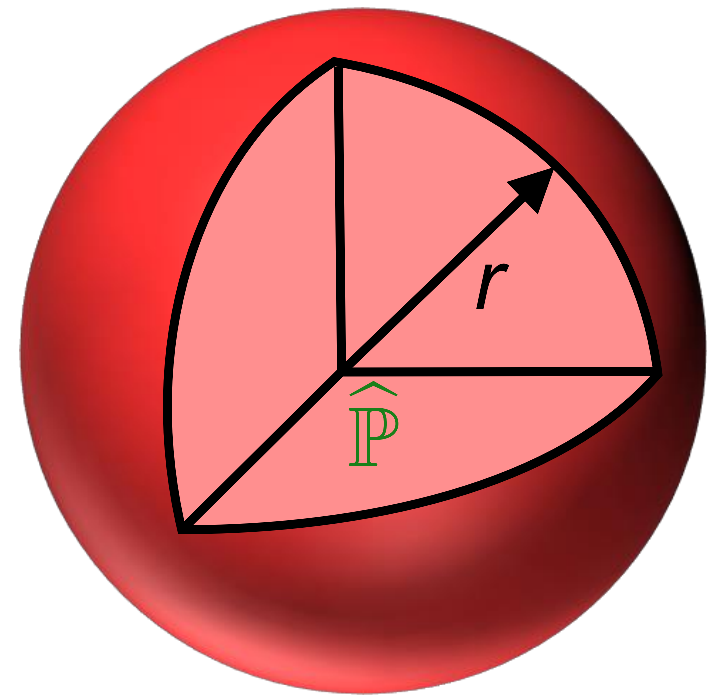
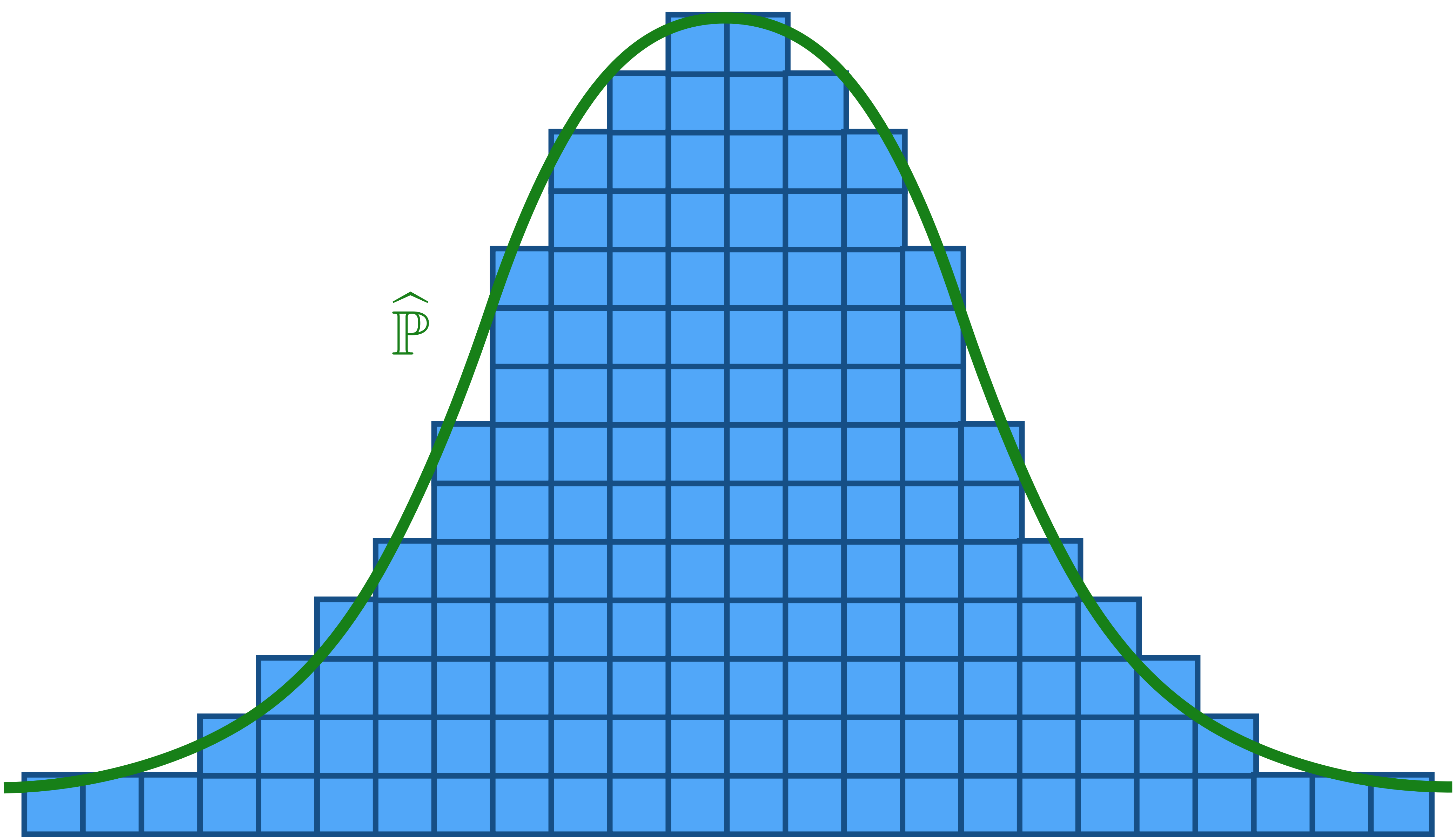


$$\mathbb{B}_r(\widehat{\mathbb{P}}) = \left\{ \mathbb{P} \text{ supported on } \Xi \;\middle|\; W_p(\mathbb{P},\widehat{\mathbb{P}}) \le r \right\}$$

[1] Pflug & Wozabal, *Quant. Finance*, 2007; Pflug & Pichler, *Springer*, 2014.

OT meets DRO



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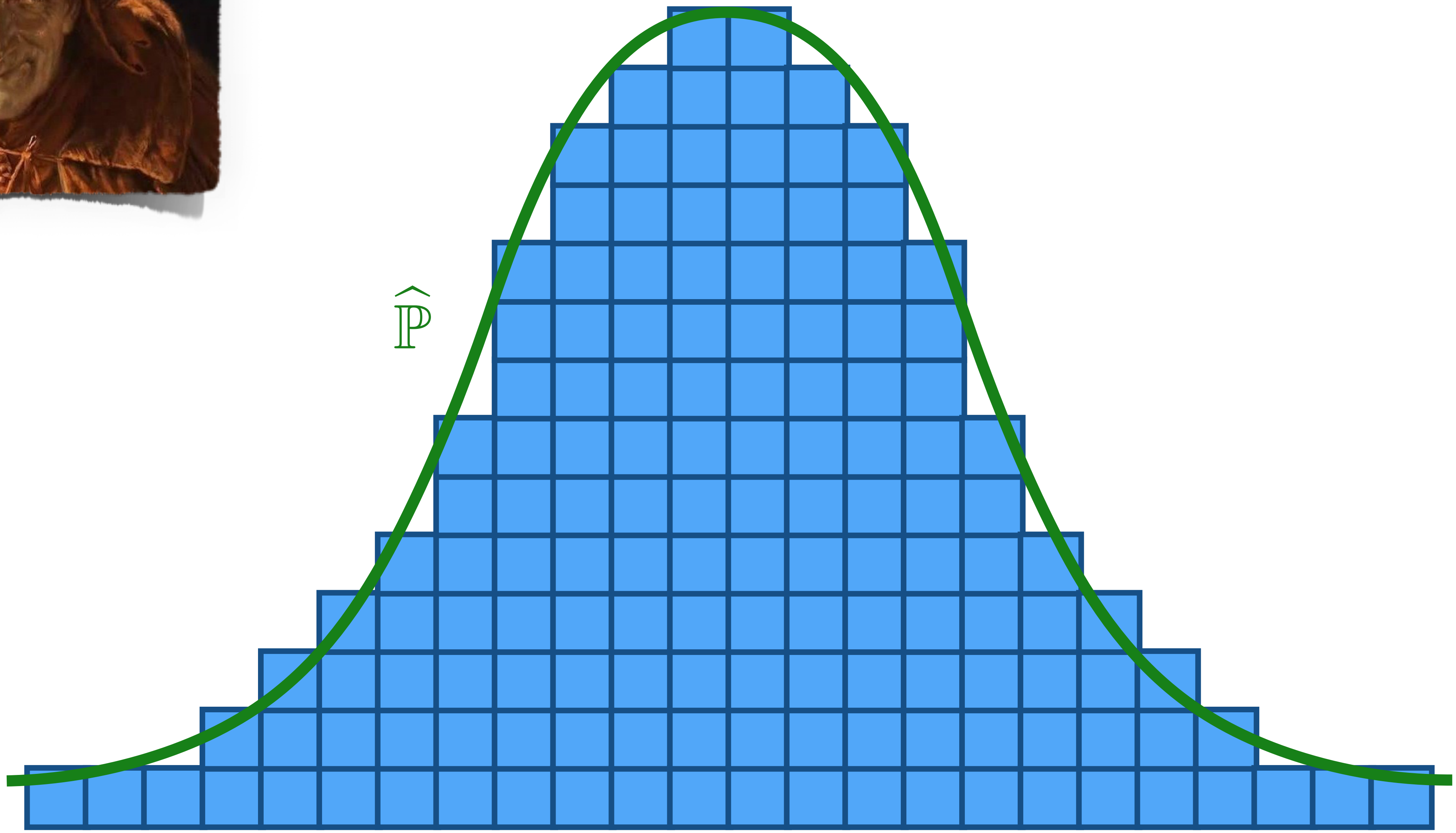

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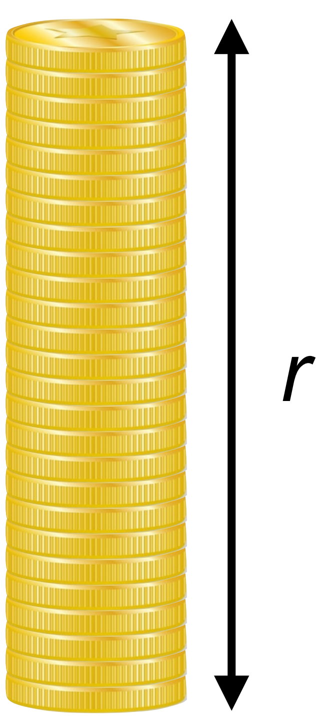

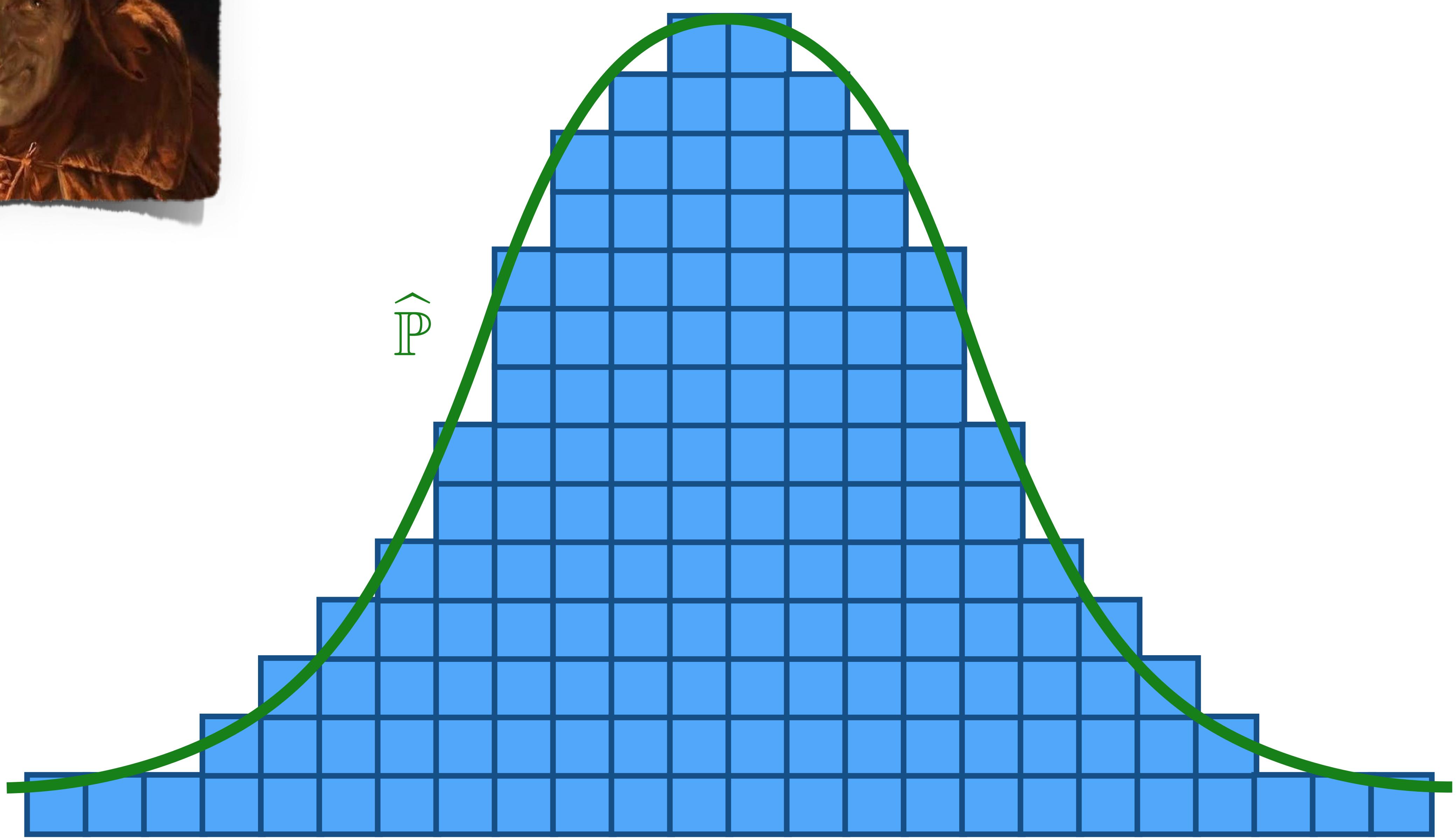

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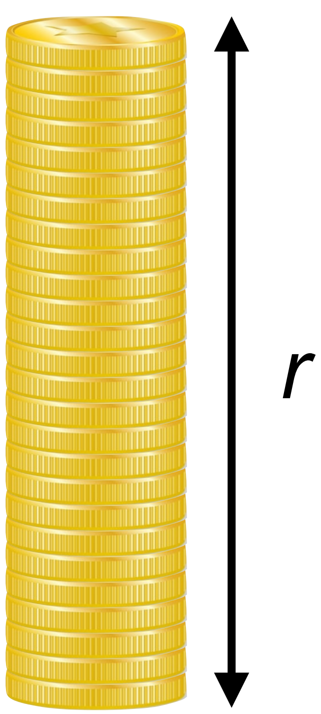

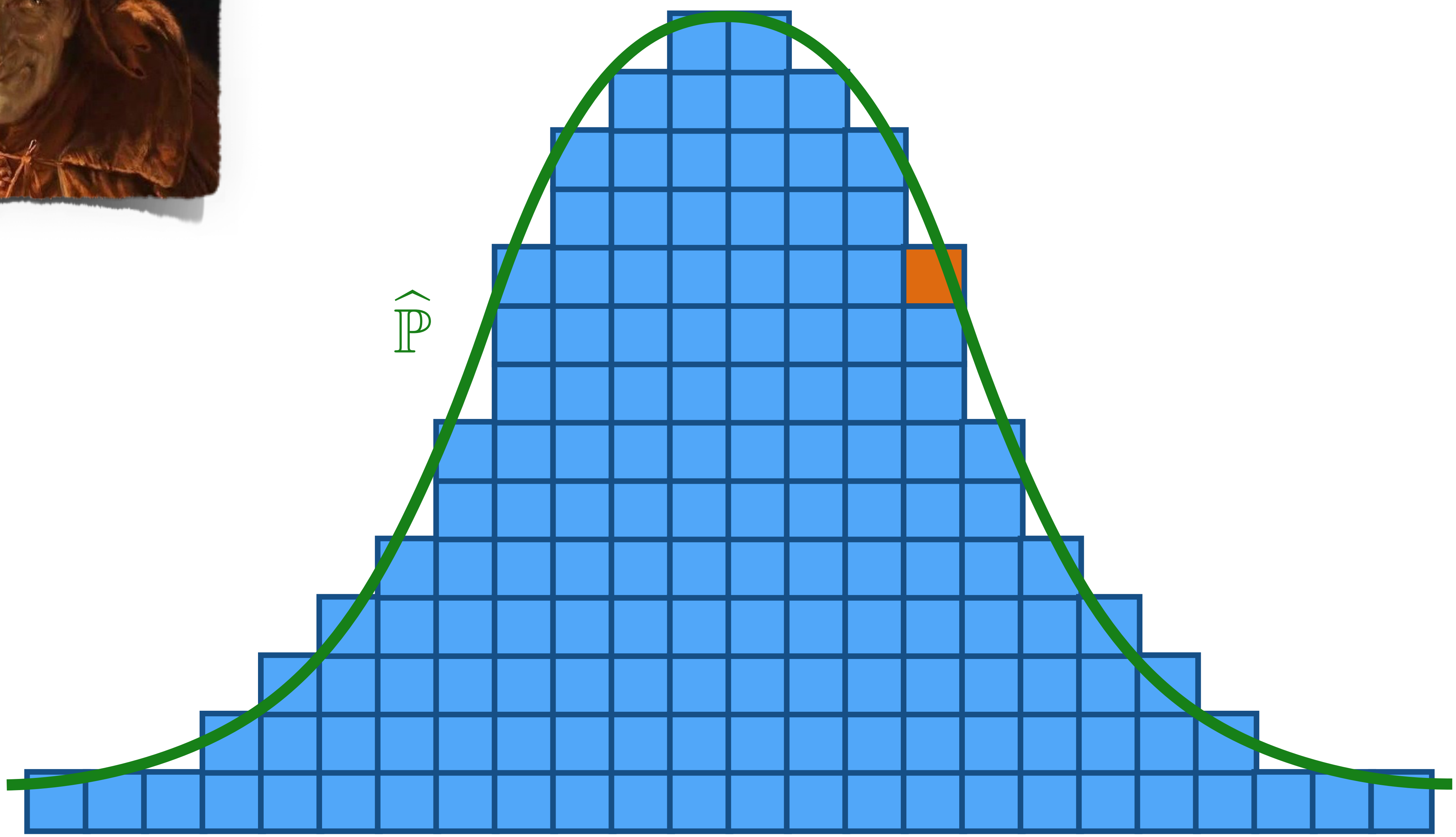

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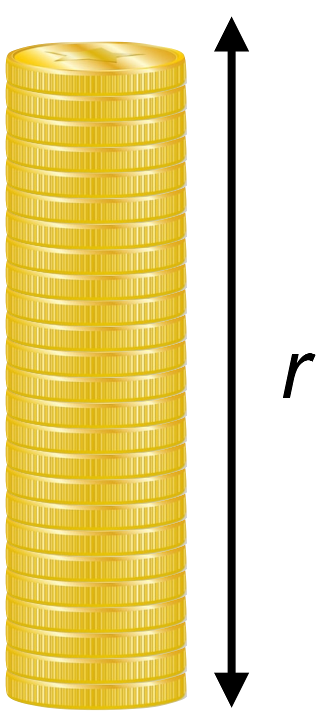

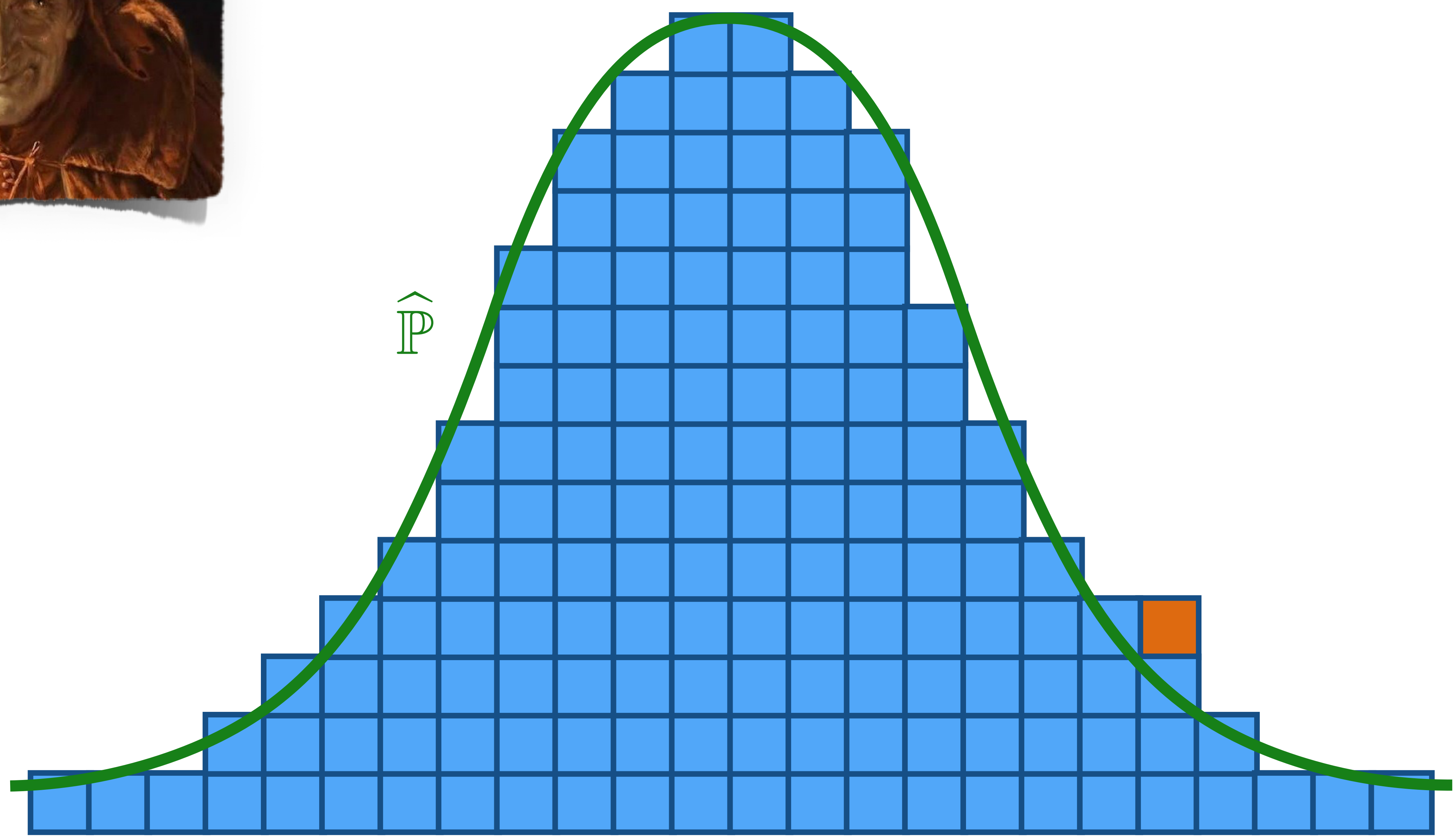

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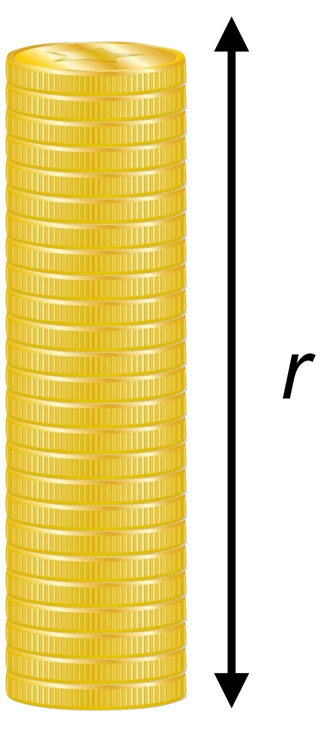
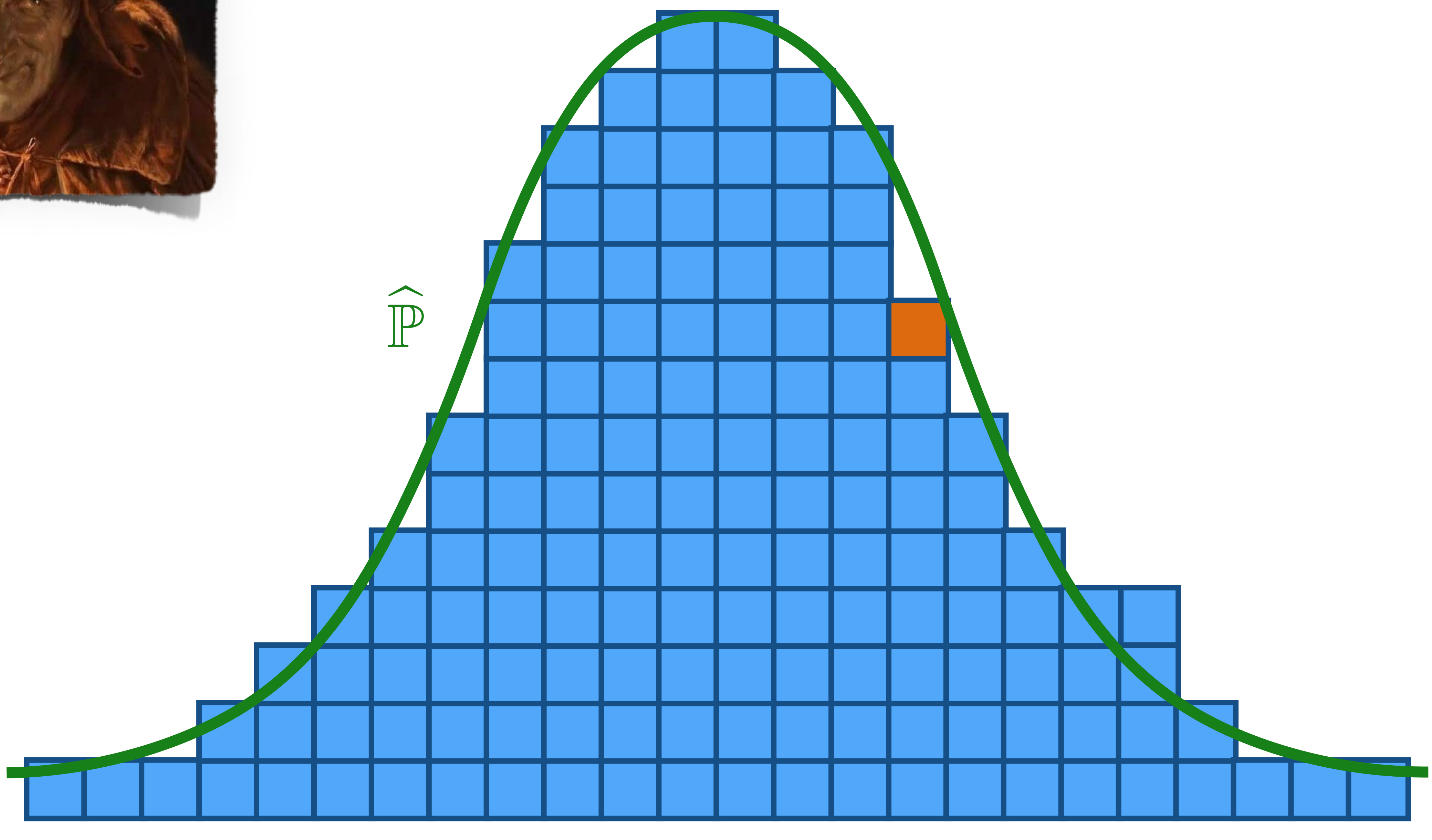
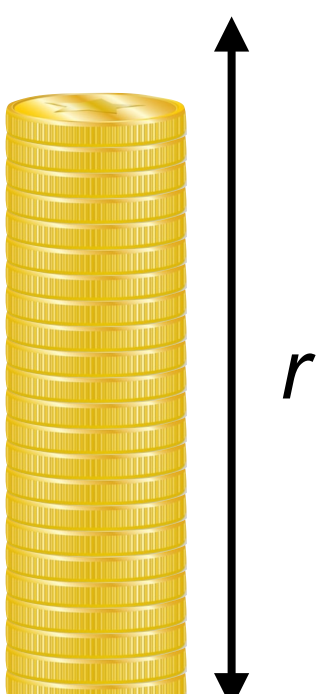
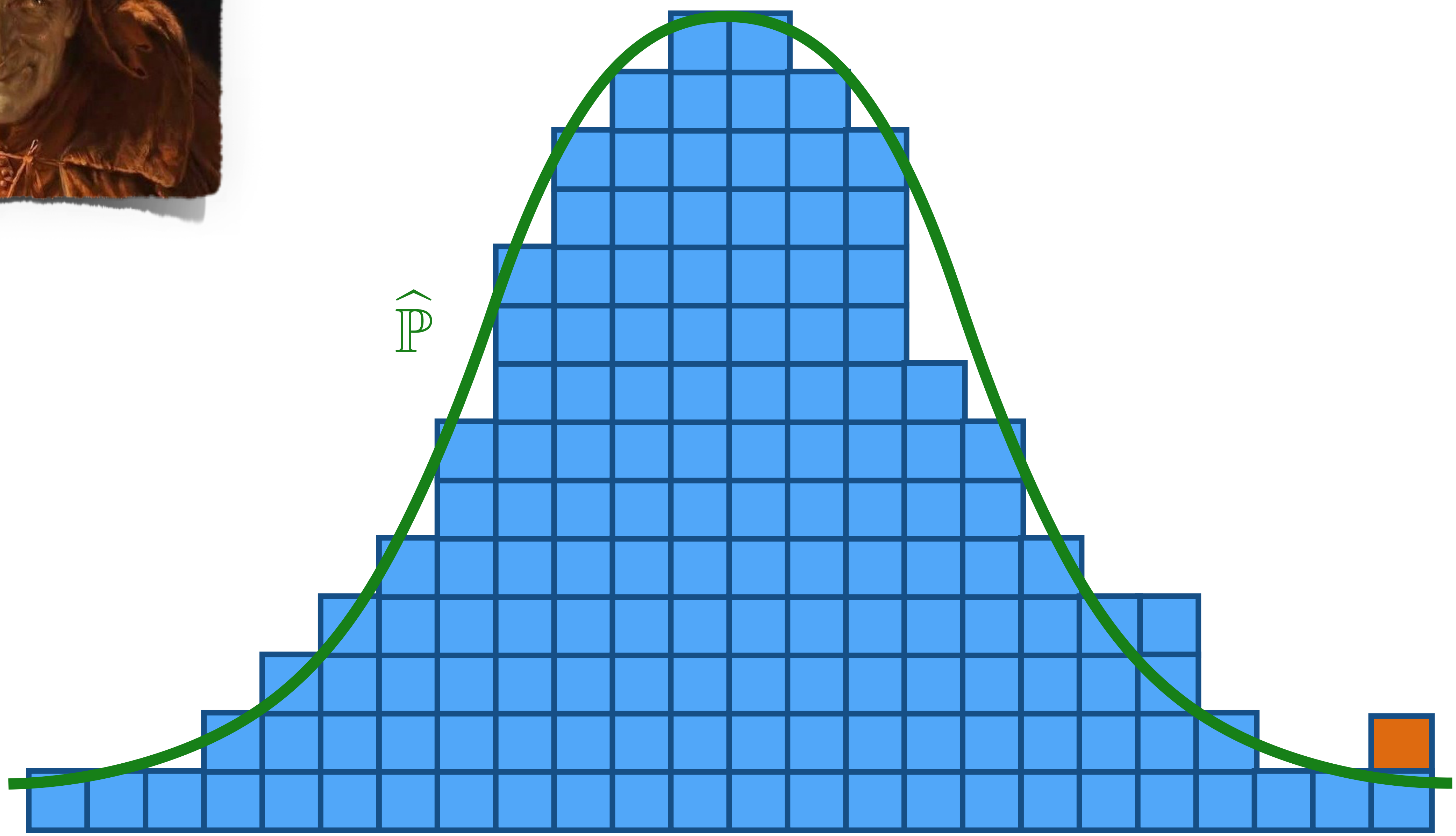


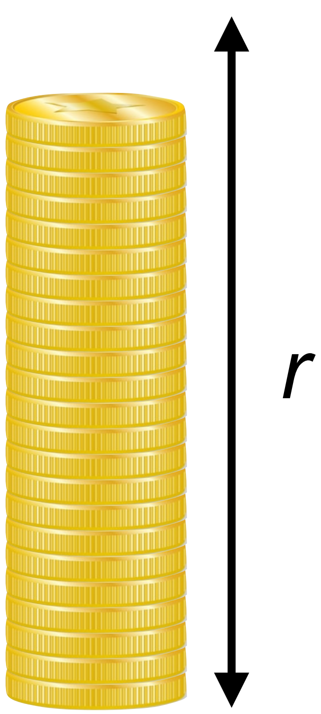


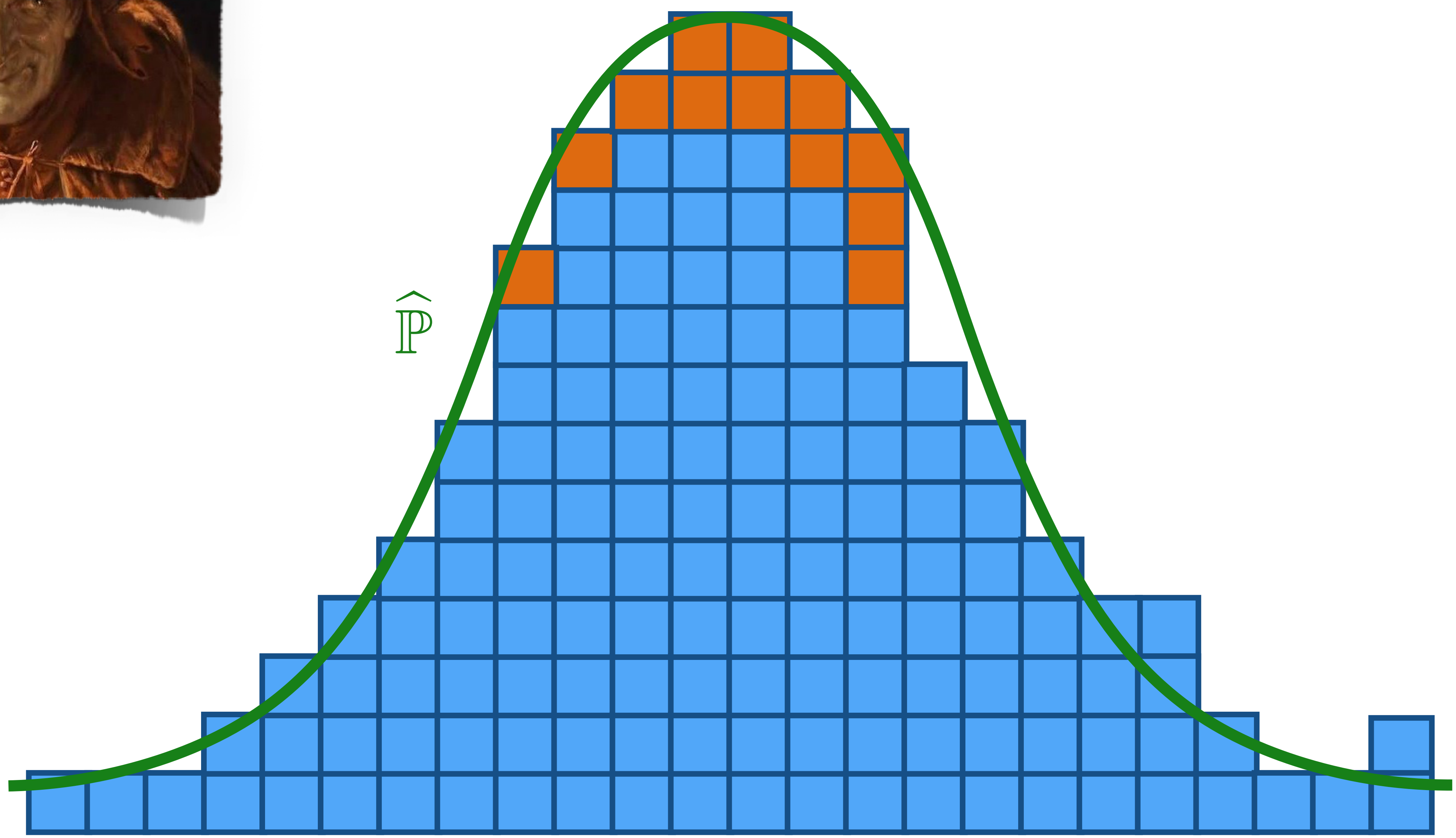
OT meets DRO



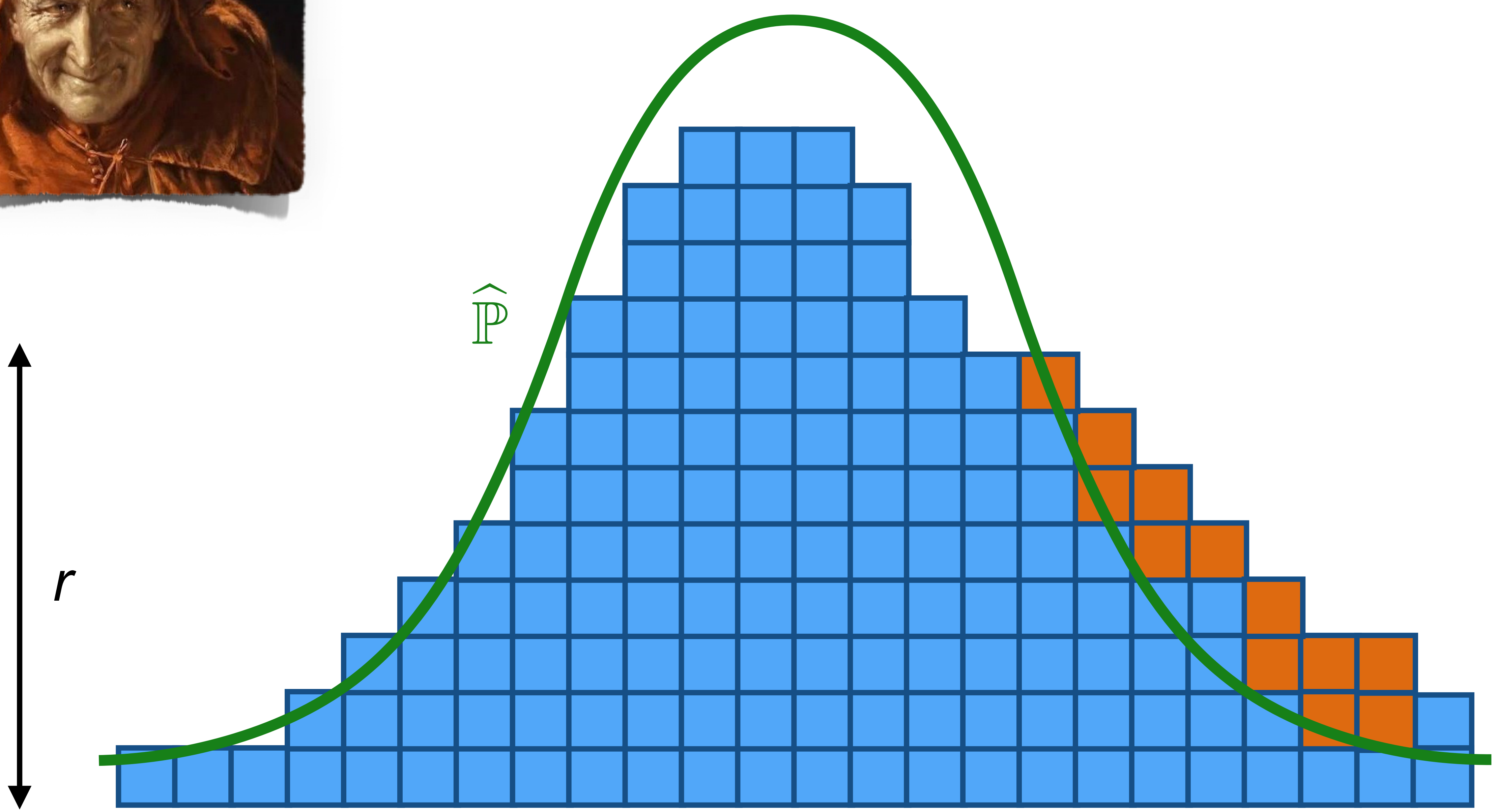


OT meets DRO

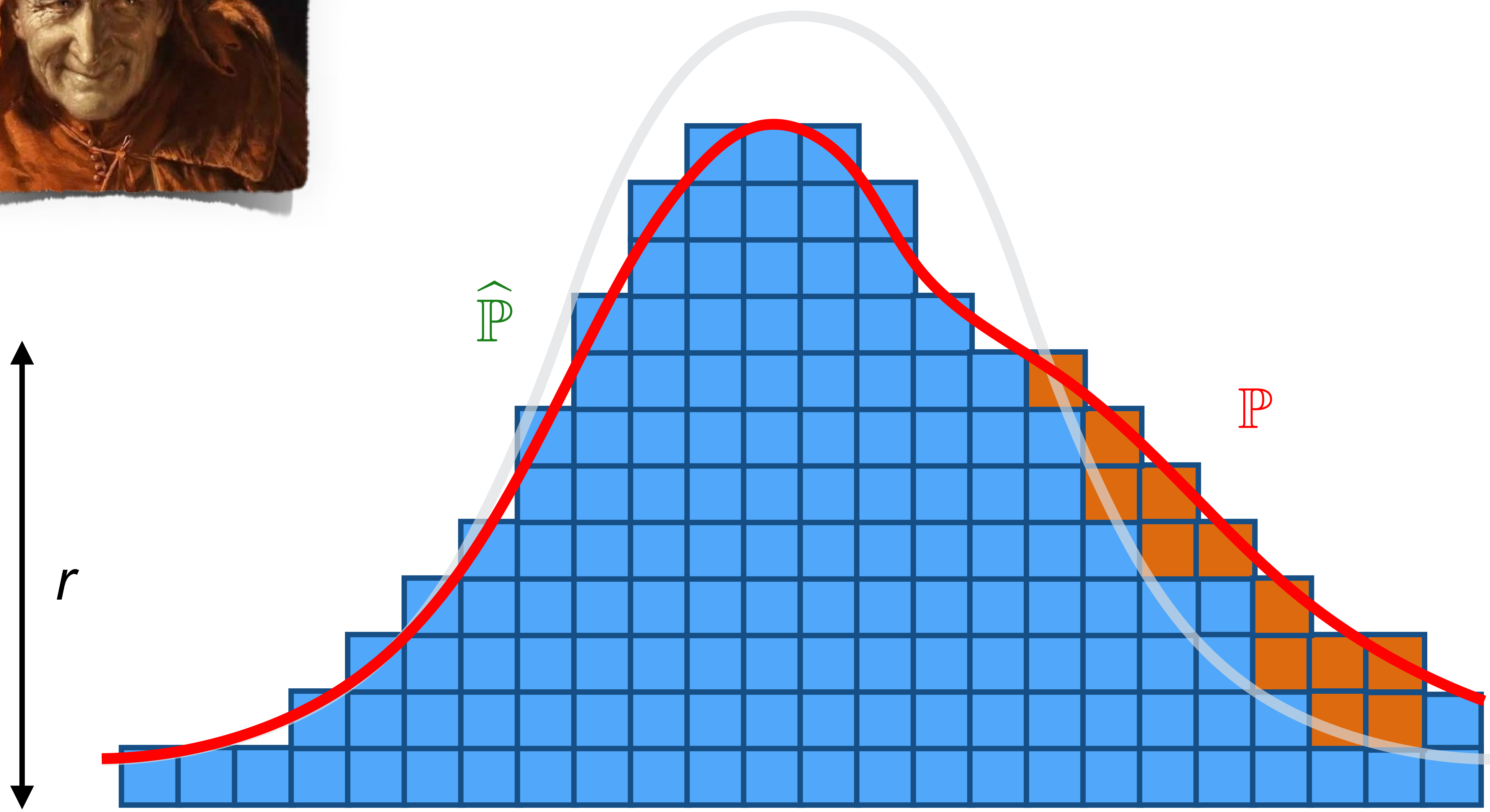




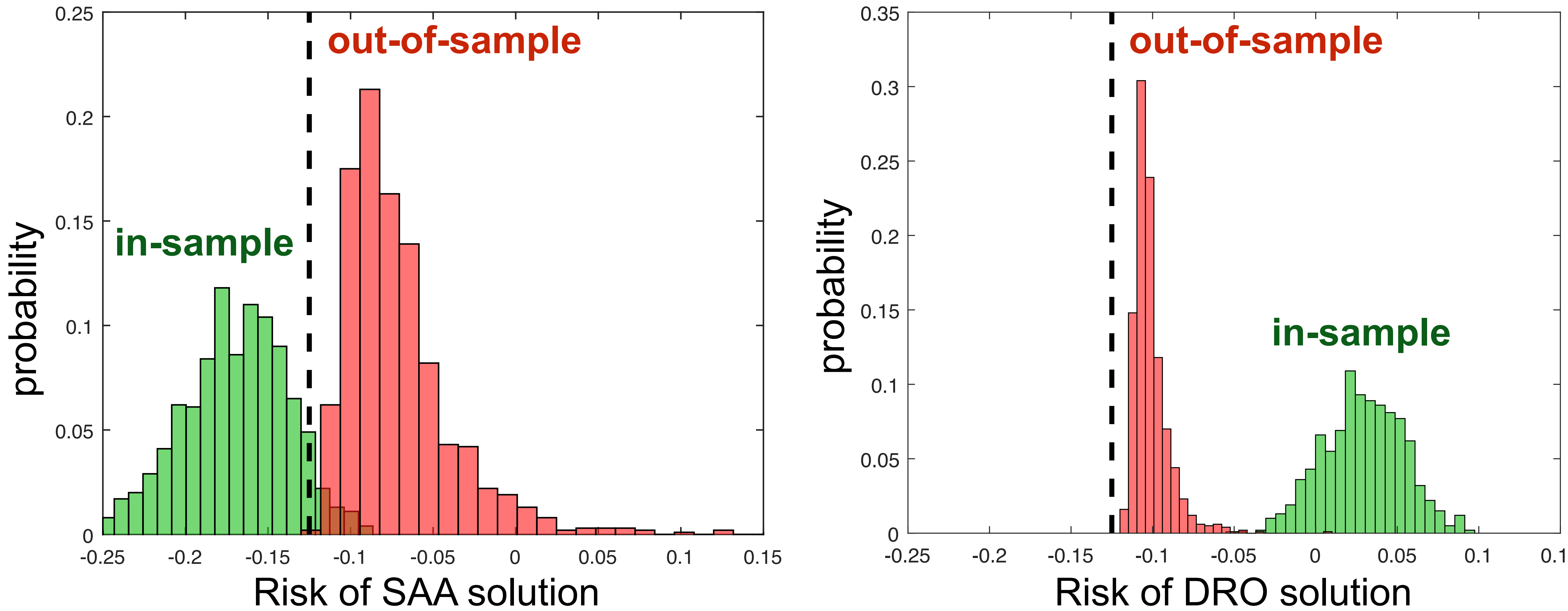
OT meets DRO



OT meets DRO

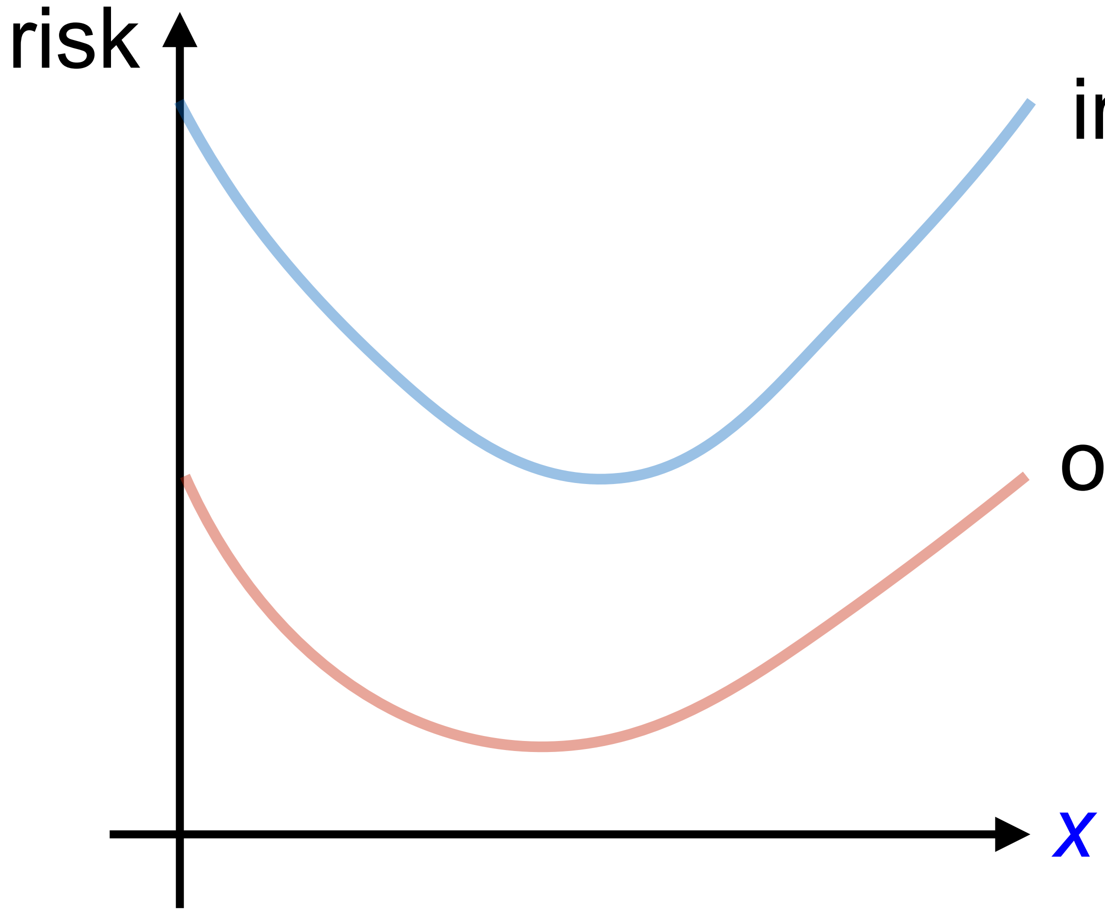


Underpromising and Overdelivering



1) Mohajerin Esfahani & Kuhn, *Math. Program.*, 2018.

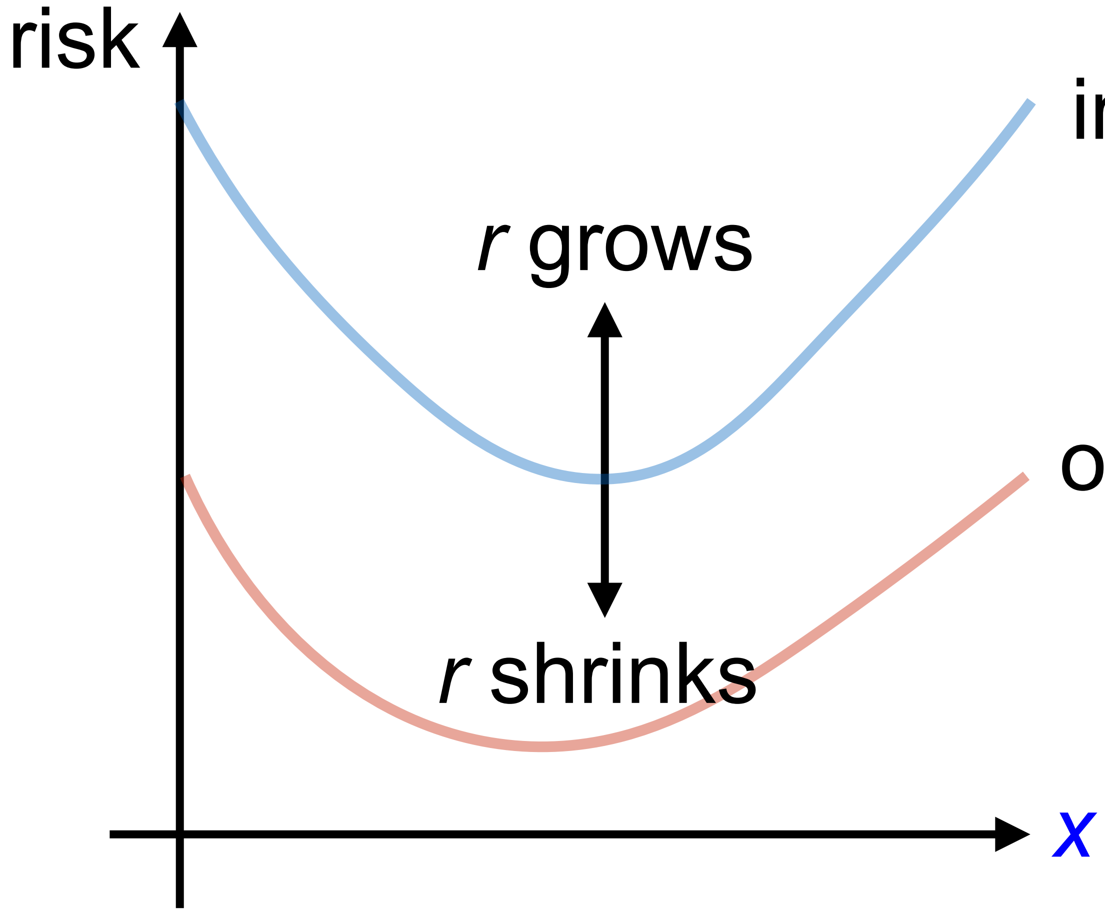
Statistical Guarantees



in-sample: $\sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)]$

out-of-sample: $\mathbb{E}_{\xi \sim \mathbb{P}^\star} [\ell(x, \xi)]$

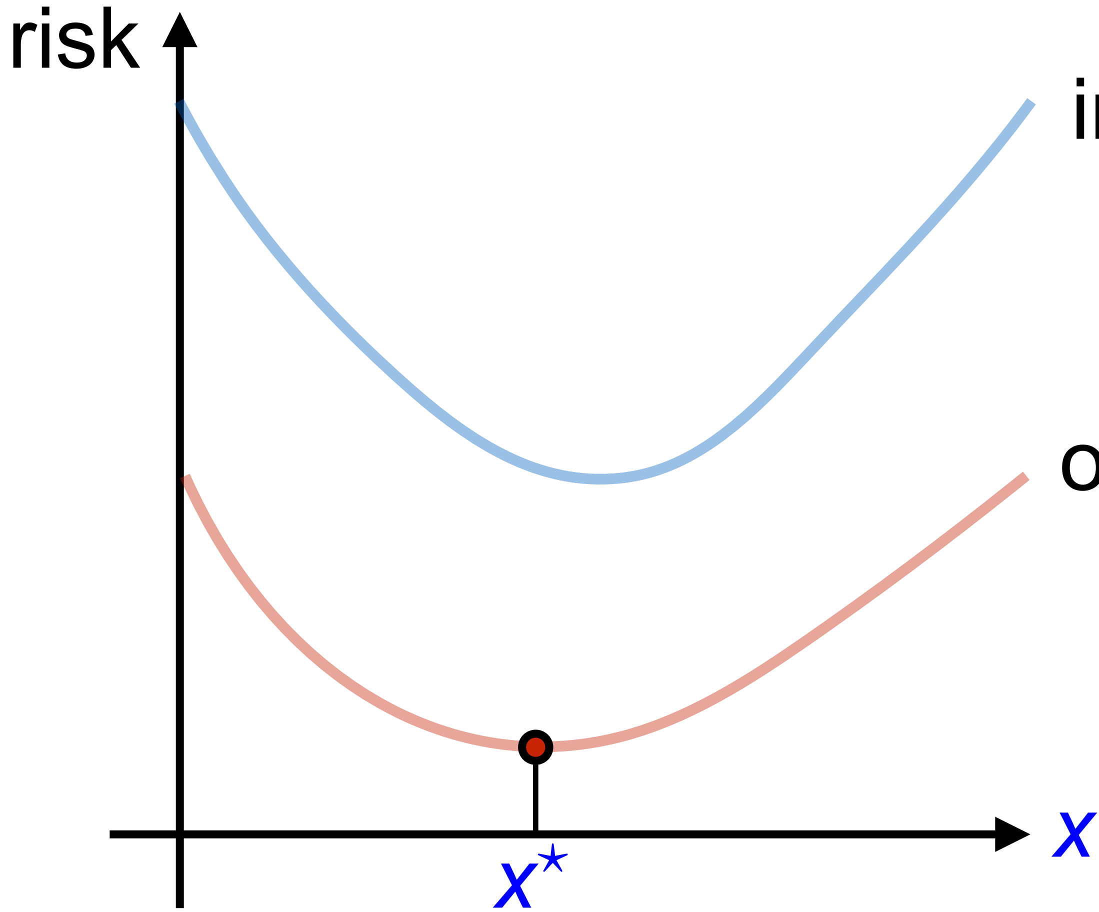
Statistical Guarantees



in-sample: $\sup_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right]$

out-of-sample: $\mathbb{E}_{\xi\sim\mathbb{P}^\star}\left[\ell(x,\xi)\right]$

Statistical Guarantees



in-sample: $\sup\limits_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right]$

out-of-sample: $\mathbb{E}_{\xi\sim\mathbb{P}^\star}\left[\ell(x,\xi)\right]$

Choice of radius:

- confidence set for[1] $\mathbb{P}^\star$: $r = O(N^{-1/d})$
- confidence set for[2] $x^\star$: $r = O(N^{-1/2})$
- confidence set for[3] $J^\star(x)$: $r = O(N^{-1/2})$

---

1) Mohajerin Esfahani & Kuhn, *Math. Program.*, 2018.

2) Blanchet, Kang & Murthy, *J. Appl. Prob.*, 2019; Blanchet & Kang, *Oper. Res.*, 2021.

3) Gao, *Oper. Res.*, 2022.

Robustification vs Regularization

Theorem:[1] If $p \geq 1$, then

$$\sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\xi \sim \mathbb{P}}[\ell(x, \xi)] \leq \begin{cases} \mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N}[\ell(x, \xi)] \\ \cdots + \sum_{k=1}^{p-1} \frac{r^k}{k!} \mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N}\left[\|D_\xi^k \ell(x, \xi)\|^{q_k}\right]^{\frac{1}{q_k}} \\ \cdots + \frac{r^p}{p!} \sup_{\xi \in \Xi} \|D_\xi^p \ell(x, \xi)\|. \end{cases}$$

[1] Shafieezadeh-Abadeh, Aolaritei, Dörfler & Kuhn, *Working Paper*, 2023.

Robustification vs Regularization

Theorem:[1] If $p \geq 1$, then

nominal loss

$$\sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)] \leq \begin{cases} \boxed{\mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N} [\ell(x, \xi)]} \\ \cdots + \sum_{k=1}^{p-1} \frac{r^k}{k!} \mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N} \left[\| D_\xi^k \ell(x, \xi) \|^{q_k} \right]^{\frac{1}{q_k}} \\ \cdots + \frac{r^p}{p!} \sup_{\xi \in \Xi} \| D_\xi^p \ell(x, \xi) \|. \end{cases}$$

---

[1] Shafieezadeh-Abadeh, Aolaritei, Dörfler & Kuhn, *Working Paper*, 2023.

Robustification vs Regularization

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variation regularization

[1] Shafieezadeh-Abadeh, Aolaritei, Dörfler & Kuhn, *Working Paper*, 2023.

Robustification vs Regularization

Theorem:[1] If $p \geq 1$, then

$$\sup_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right] \leq \begin{cases} \mathbb{E}_{\xi\sim\widehat{\mathbb{P}}_N}\left[\ell(x,\xi)\right] \\ \cdots + \displaystyle\sum_{k=1}^{p-1} \frac{r^k}{k!} \mathbb{E}_{\xi\sim\widehat{\mathbb{P}}_N}\left[\|D_\xi^k \ell(x,\xi)\|^{q_k}\right]^{\frac{1}{q_k}} \\ \cdots + \boxed{\dfrac{r^p}{p!} \displaystyle\sup_{\xi\in\Xi} \|D_\xi^p \ell(x,\xi)\|}. \end{cases}$$

Lipschitz regularization

[1] Shafieezadeh-Abadeh, Aolaritei, Dörfler & Kuhn, *Working Paper*, 2023.

Robustification vs Regularization

Theorem:[1] If $p \geq 1$, then

$$\sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)] \leq \begin{cases} \mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N} [\ell(x, \xi)] \\ \cdots + \sum_{k=1}^{p-1} \frac{r^k}{k!} \mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N} \left[\| D_\xi^k \ell(x, \xi) \|^{q_k} \right]^{\frac{1}{q_k}} \\ \cdots + \frac{r^p}{p!} \sup_{\xi \in \Xi} \| D_\xi^p \ell(x, \xi) \|. \end{cases}$$

Higher-order variation regularization used for:

- adversarial training of NNs (*e.g.*, Hein & Andriushchenko, *NeurIPS,* 2017)
- stabilizing training of GANs (*e.g.*, Roth *et al.*, *NeurIPS* 2017)
- regularization in imaging (*e.g.*, Bredies *et al.*, *SIAM J. Imaging Sci.*, 2010)

[1] Shafieezadeh-Abadeh, Aolaritei, Dörfler & Kuhn, *Working Paper*, 2023.

Robustification vs Regularization

Theorem:[1] If $p = 1$, then

$$\sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)] \leq \mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N} [\ell(x, \xi)] + r \, \mathrm{lip}(\ell(x, \cdot)).$$

[1] Shafieezadeh-Abadeh, Aolaritei, Dörfler & Kuhn, *Working Paper*, 2023.

Robustification vs Regularization

Theorem:[1] If $p = 1$, ℓ is convex in ξ and $\Xi = \mathbb{R}^d$, then

$$\sup_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right] = \mathbb{E}_{\xi\sim\widehat{\mathbb{P}}_N}\left[\ell(x,\xi)\right] + r\,\mathrm{lip}(\ell(x,\cdot)).$$

[1] Mohajerin Esfahani & Kuhn, *Math. Program.*, 2018.

Robustification vs Regularization

Theorem:[1] If $p = 1$, ℓ is univariate convex and $\Xi = \mathbb{R}^d$, then

$$\sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}}_N)} \mathbb{E}_{\xi \sim \mathbb{P}} \left[\ell(x^\top \xi) \right] = \mathbb{E}_{\xi \sim \widehat{\mathbb{P}}_N} \left[\ell(x^\top \xi) \right] + r \, \mathrm{lip}(\ell) \, \|x\|_* .$$

[1] Shafieeezadeh-Abadeh, Mohajerin Esfahani & Kuhn, *NeurIPS*, 2015; *JMLR* 2019.

Robustification vs Regularization

Theorem:[1] If $p = 1$, ℓ is univariate convex and $\Xi = \mathbb{R}^d$, then

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Offers robustness interpretation for:

- lasso regularization
- basis pursuit denoising
- regularized least absolute deviation regression
- the Dantzig selector

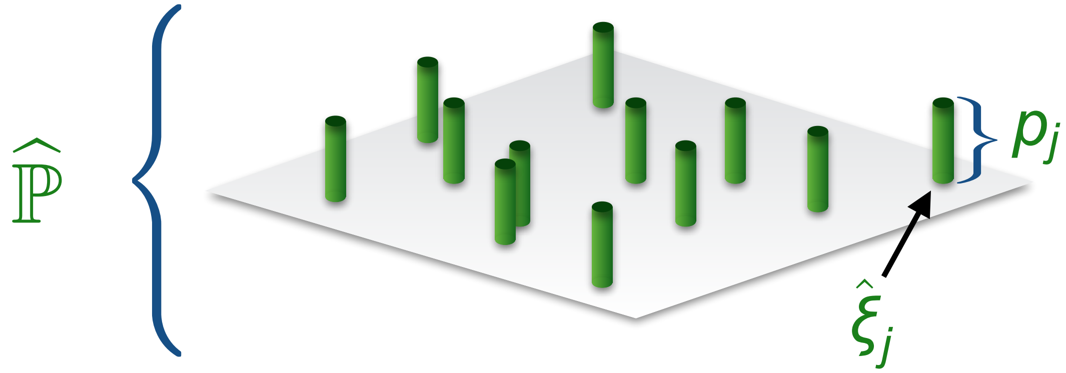
[1] Shafieeezadeh-Abadeh, Mohajerin Esfahani & Kuhn, *NeurIPS*, 2015; *JMLR* 2019.

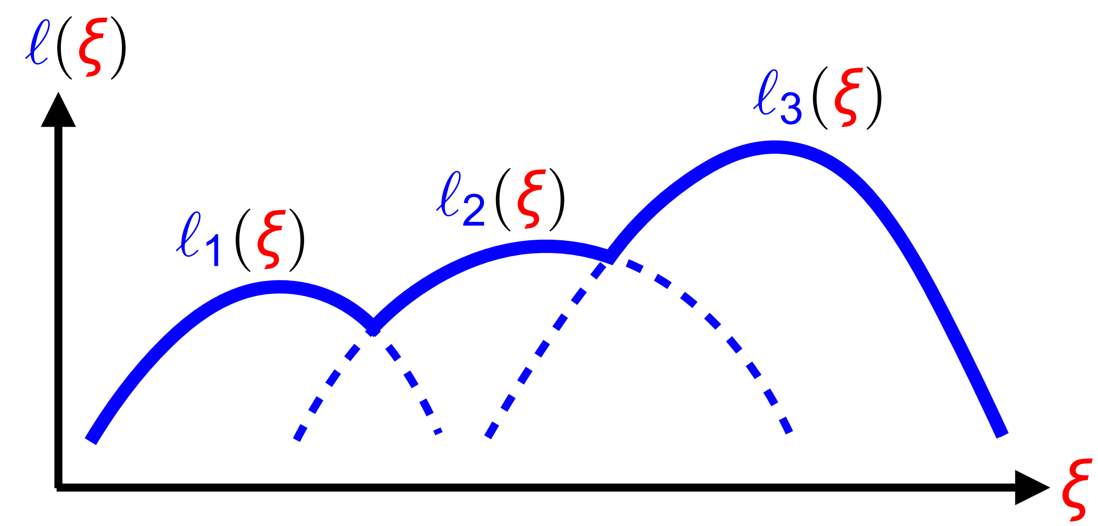
Computational Tractability

Primal DRO problem:

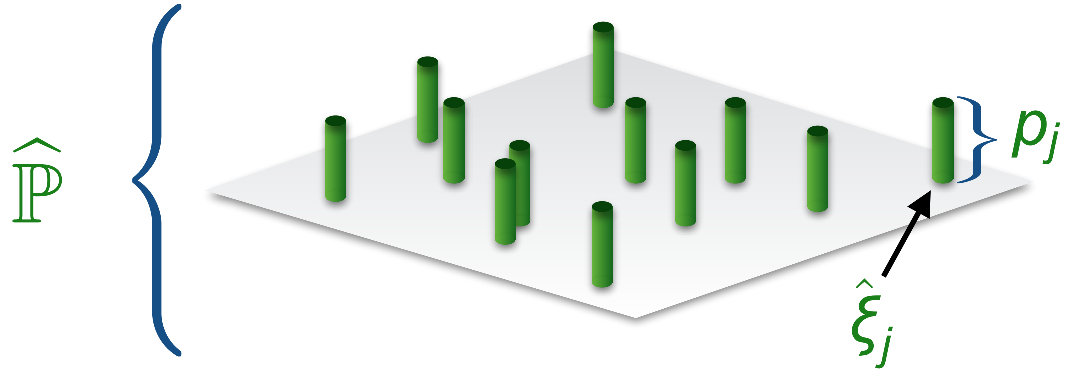
$$\inf_{x \in \mathcal{X}} \sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} \left[\ell(x, \xi) \right]$$

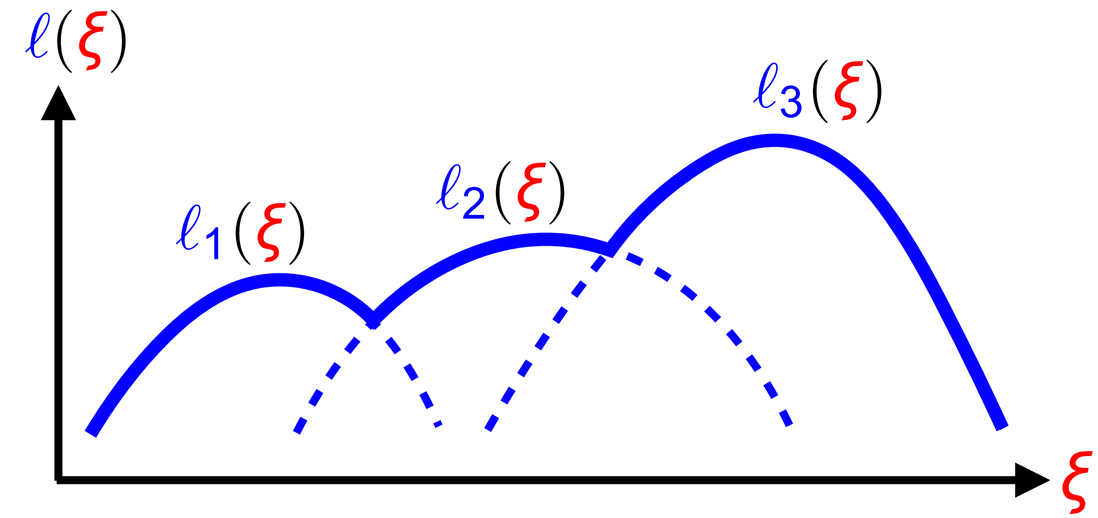
Computational Tractability





Computational Tractability





Theorem:[1] The **primal** DRO problem is equivalent to

$$
\begin{array}{lll}
\inf & \lambda r + \sum_j p_j s_j & \\
\text{s.t.} & x \in \mathcal{X},\ \lambda, \tau_{ij} \in \mathbb{R}_+,\ s_j \in \mathbb{R},\ \zeta_{ij}^{\ell}, \zeta_{ij}^{c}, \zeta_{ij}^{\Xi} \in \mathbb{R}^d & \forall i,j \\
& (-\ell_i)^{*2}(x, \zeta_{ij}^{\ell}) + \lambda c^{*1}(\zeta_{ij}^{c}/\lambda, \hat{\xi}_j) + \tau_{ij}\sigma_{\Xi}(\zeta_{ij}^{\Xi}/\tau_{ij}) \le s_j & \forall i,j \\
& \zeta_{ij}^{\ell} + \zeta_{ij}^{c} + \zeta_{ij}^{\Xi} = 0 & \forall i,j
\end{array}
$$

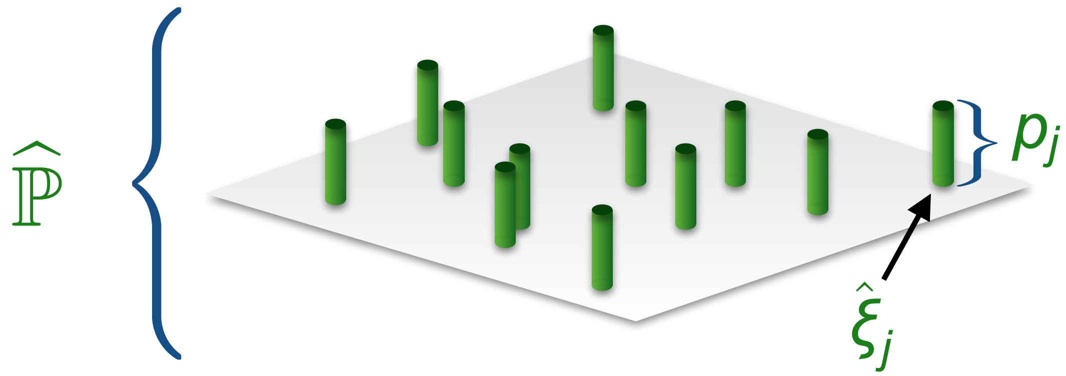
[1] Mohajerin Esfahani & Kuhn, *Math. Program.*, 2018; Zhao & Guan, *Oper. Res. Lett.*, 2018; Blanchet & Murthy, *Math. Oper. Res.*, 2019; Gao & Kleytweg, *Math. Oper. Res.*, 2022; Chen, Kuhn & Wiesemann, *Oper. Res.*, 2023.

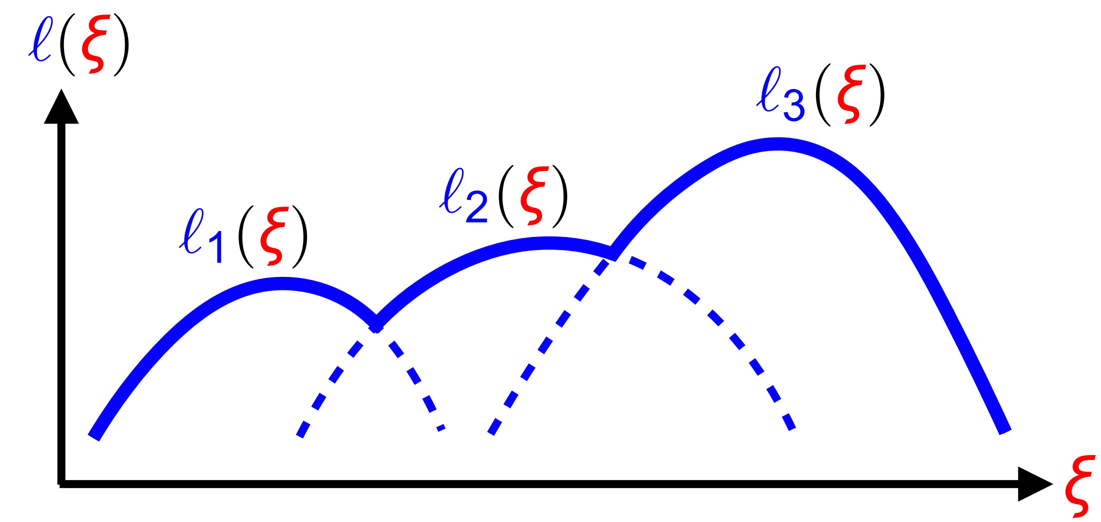
Computational Tractability

Dual DRO problem:

$$\sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \ \inf_{x \in \mathcal{X}} \ \mathbb{E}_{\xi \sim \mathbb{P}} \left[\ell(x, \xi) \right]$$

Computational Tractability





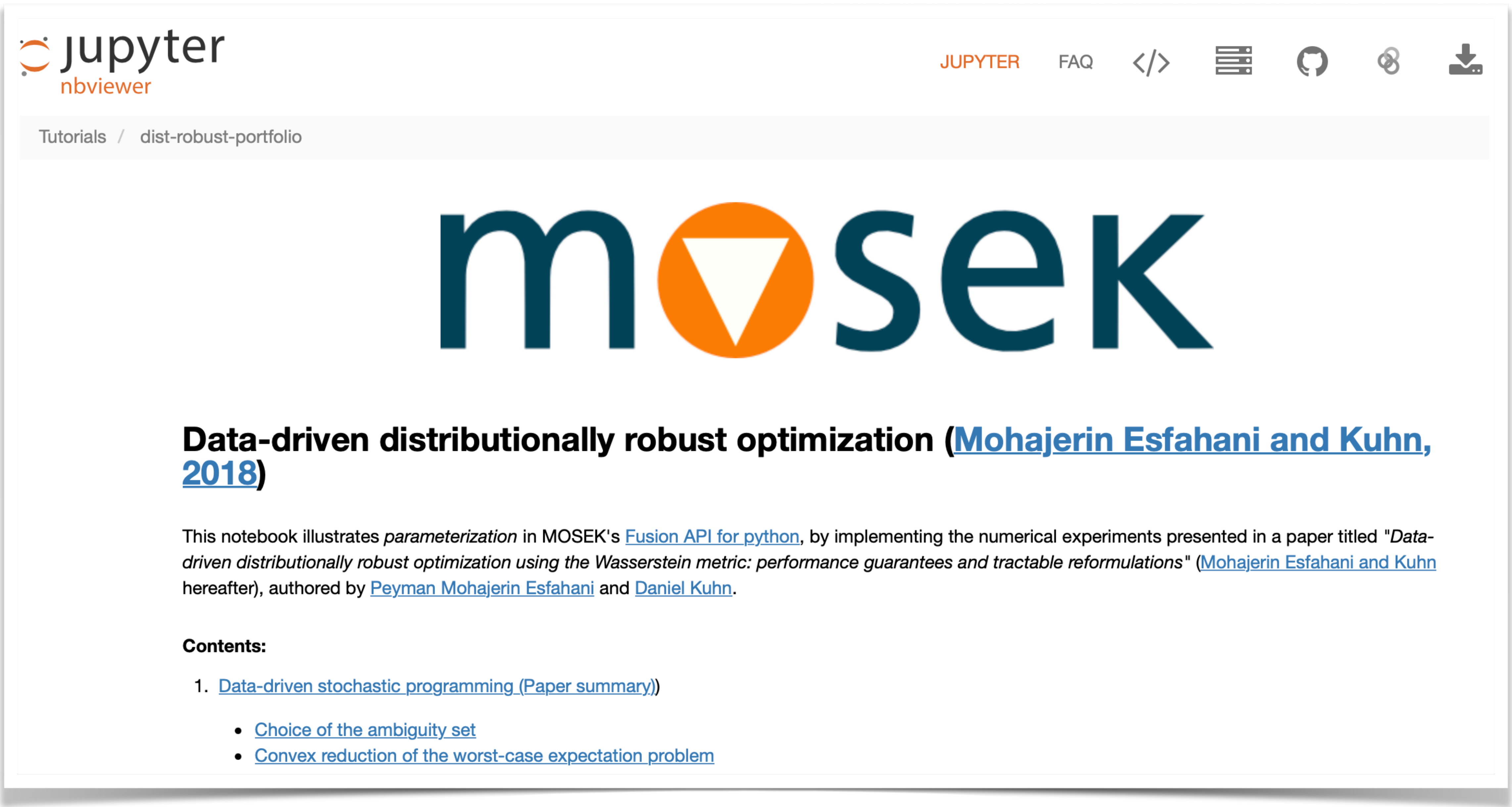
Theorem:[1] The **dual** DRO problem is equivalent to

$$
\begin{array}{lll}
\sup & -\sum_i \sum_j q_{ij} \ell_i^{*1}(\alpha_{ij}, \hat{\xi}_j + \xi_{ij}/q_{ij}) - v\, \sigma_{\mathcal{X}}(\beta/v) & \\
\text{s.t.} & q_{ij}, v \in \mathbb{R}_+, \ \xi_{ij} \in \mathbb{R}^d, \ \alpha_{ij}, \beta \in \mathbb{R}^n & \forall i, j \\
& q_{ij}\, \delta_{\Xi}(\hat{\xi}_j + \xi_{ij}/q_{ij}) \leq 0 & \forall i, j \\
& \sum_i \sum_j \alpha_{ij} + \beta = 0, \ \sum_i q_{ij} = p_j & \forall j \\
& \sum_i \sum_j q_{ij}\, c(\hat{\xi}_j + \xi_{ij}/q_{ij}, \hat{\xi}_j) \leq r &
\end{array}
$$

[1] Shafieezadeh-Abadeh, Aolaritei, Dörfler & Kuhn, *Working Paper*, 2023.

Computational Tractability

Jupyter notebook based on Mosek's Fusion API for Python:[1)]



jupyter nbviewer

JUPYTER FAQ

Tutorials / dist-robust-portfolio

Data-driven distributionally robust optimization (Mohajerin Esfahani and Kuhn, 2018)

This notebook illustrates *parameterization* in MOSEK's Fusion API for python, by implementing the numerical experiments presented in a paper titled "*Data-driven distributionally robust optimization using the Wasserstein metric: performance guarantees and tractable reformulations*" (Mohajerin Esfahani and Kuhn hereafter), authored by Peyman Mohajerin Esfahani and Daniel Kuhn.

Contents:

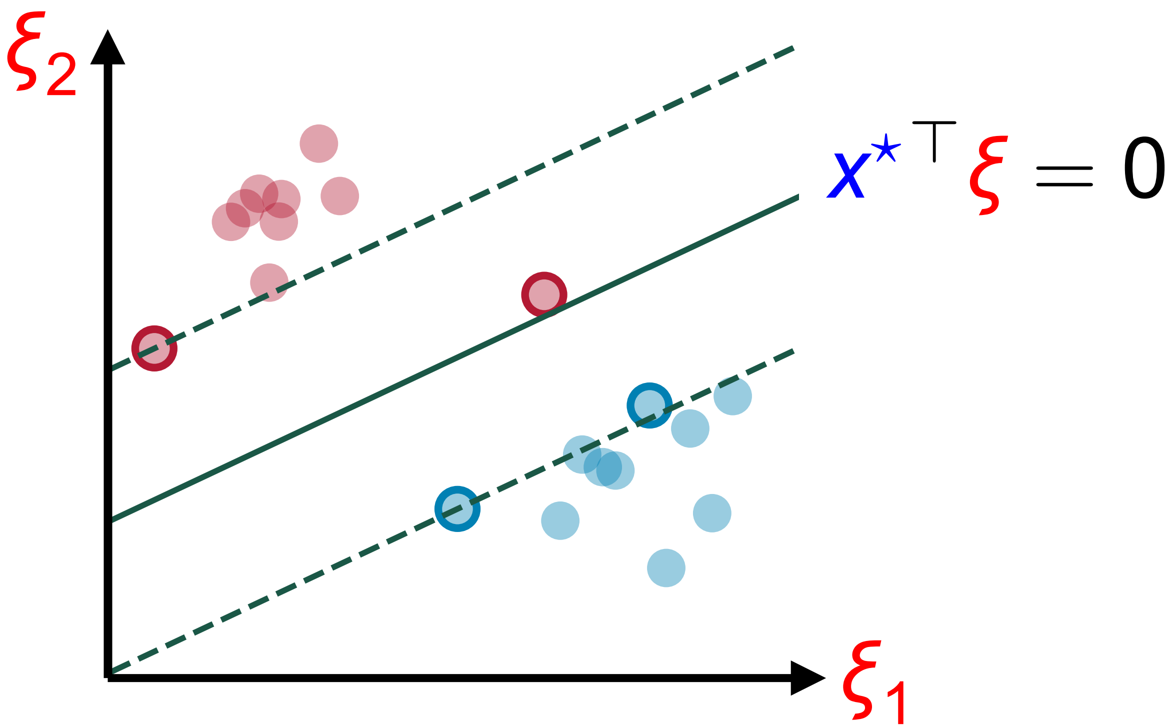
1. Data-driven stochastic programming (Paper summary))
   - Choice of the ambiguity set
   - Convex reduction of the worst-case expectation problem

---

1) https://nbviewer.jupyter.org/github/MOSEK/Tutorials

Distributionally Robust SVM

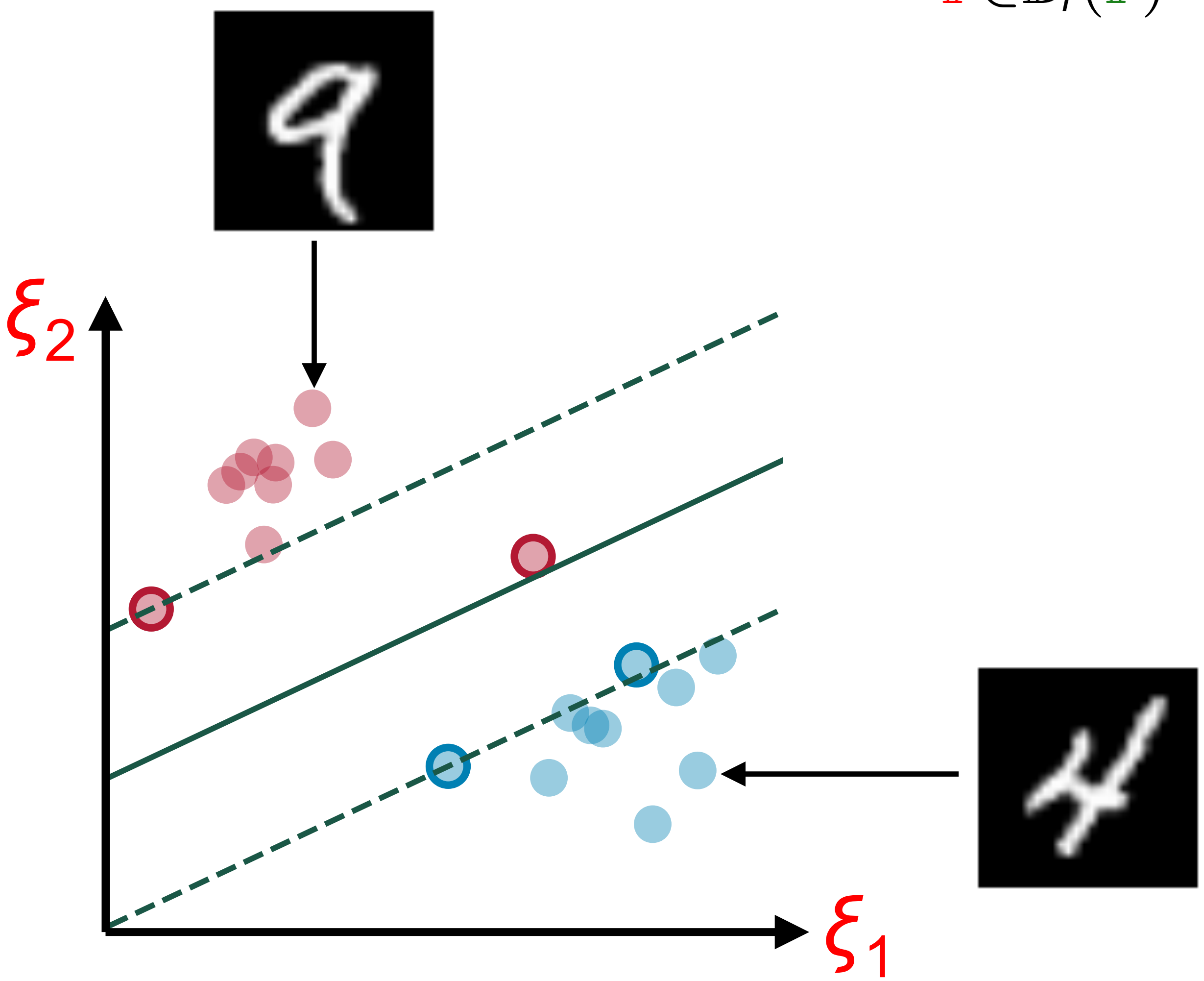
$$\inf_{x \in \mathcal{X}} \sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)]$$



empirical distribution $\widehat{\mathbb{P}}$

Distributionally Robust SVM

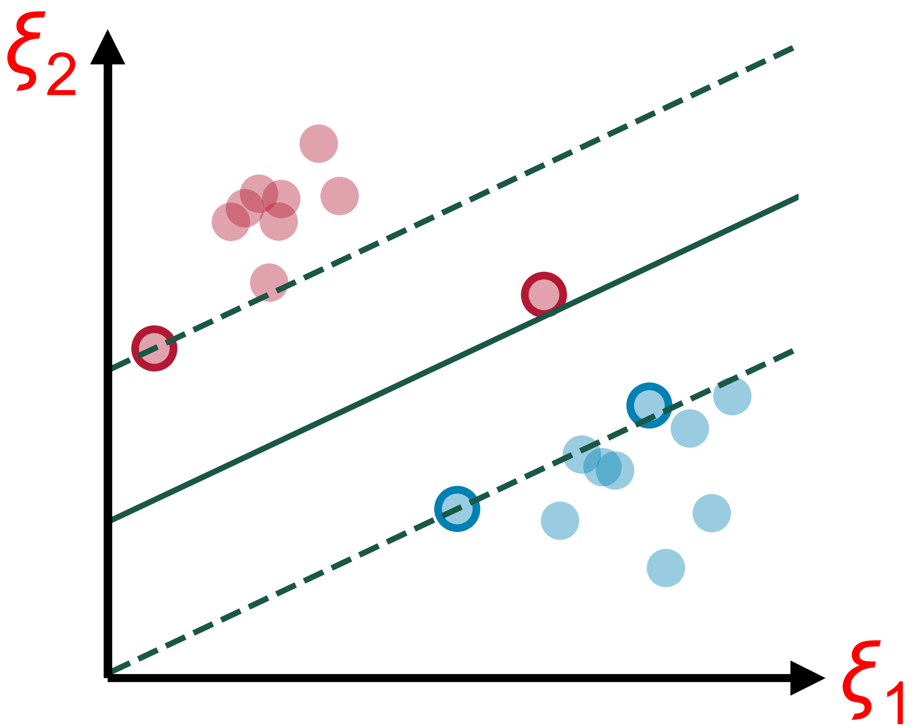
$$\inf_{x \in \mathcal{X}} \sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)]$$



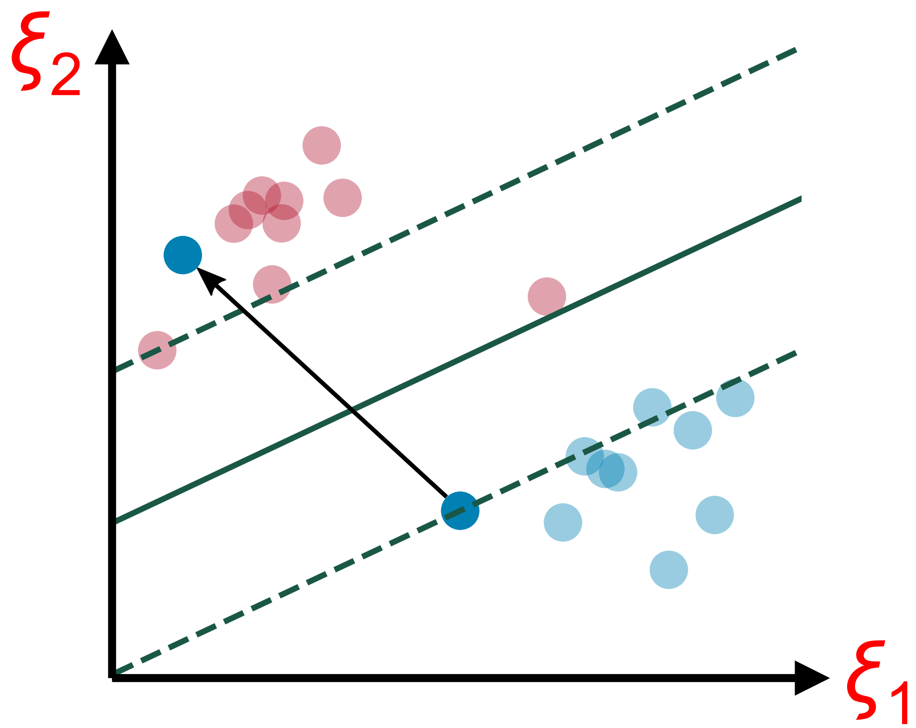
empirical distribution $\widehat{\mathbb{P}}$

Distributionally Robust SVM

$$\inf_{x \in \mathcal{X}} \sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} \left[\ell(x, \xi) \right]$$



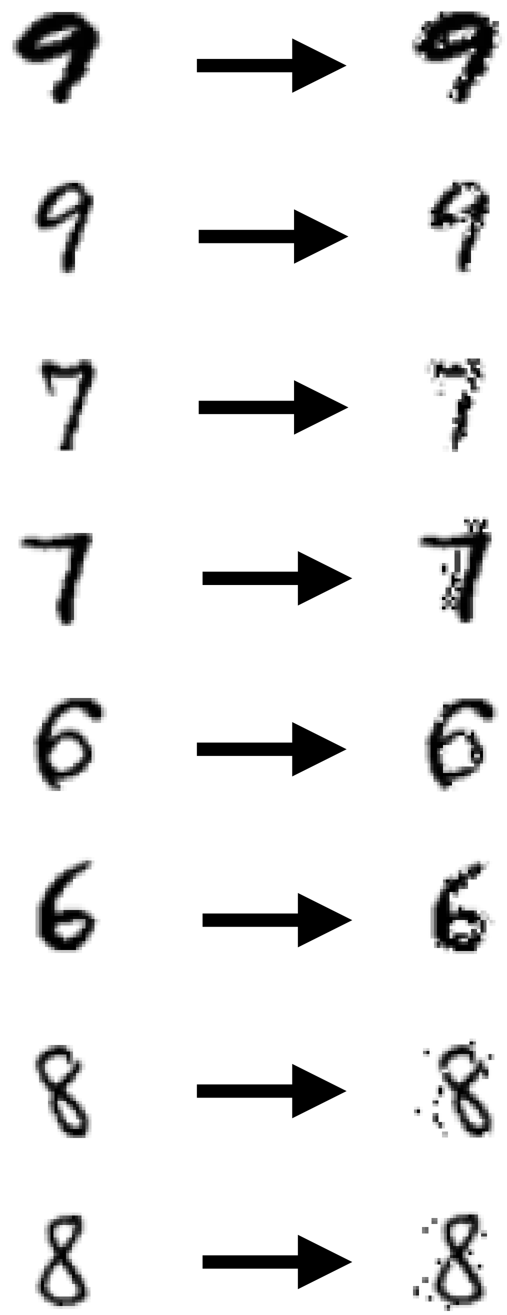
empirical distribution $\widehat{\mathbb{P}}$



worst-case distribution $\mathbb{P}^\star$

Adversarial Examples

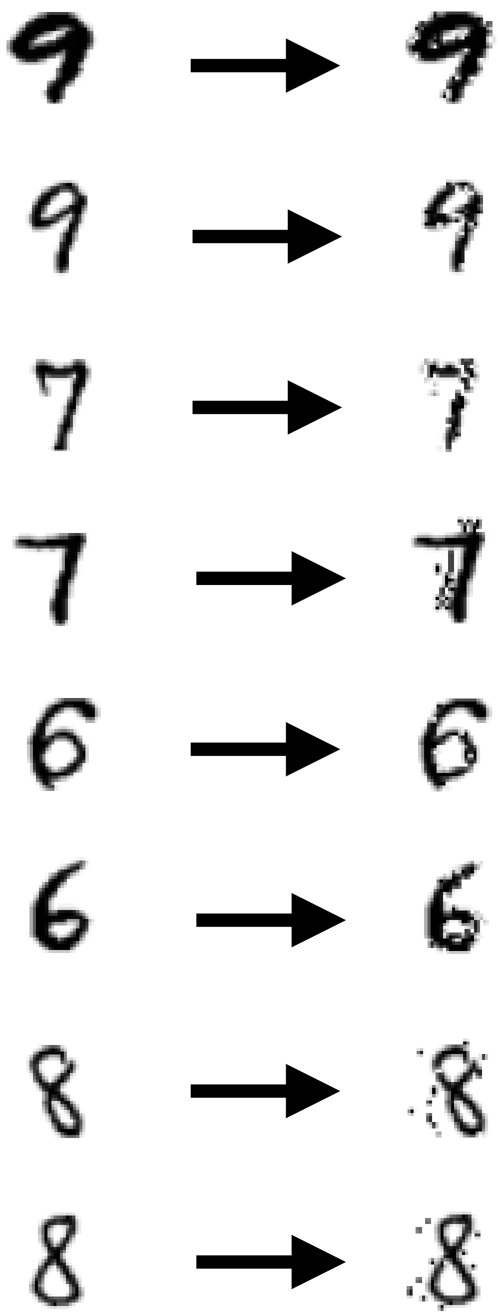
$$\mathbb{P}^\star \in \underset{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})}{\text{argmax}} \; \mathbb{E}_{\xi \sim \mathbb{P}} \left[\ell(x^\star, \xi) \right]$$

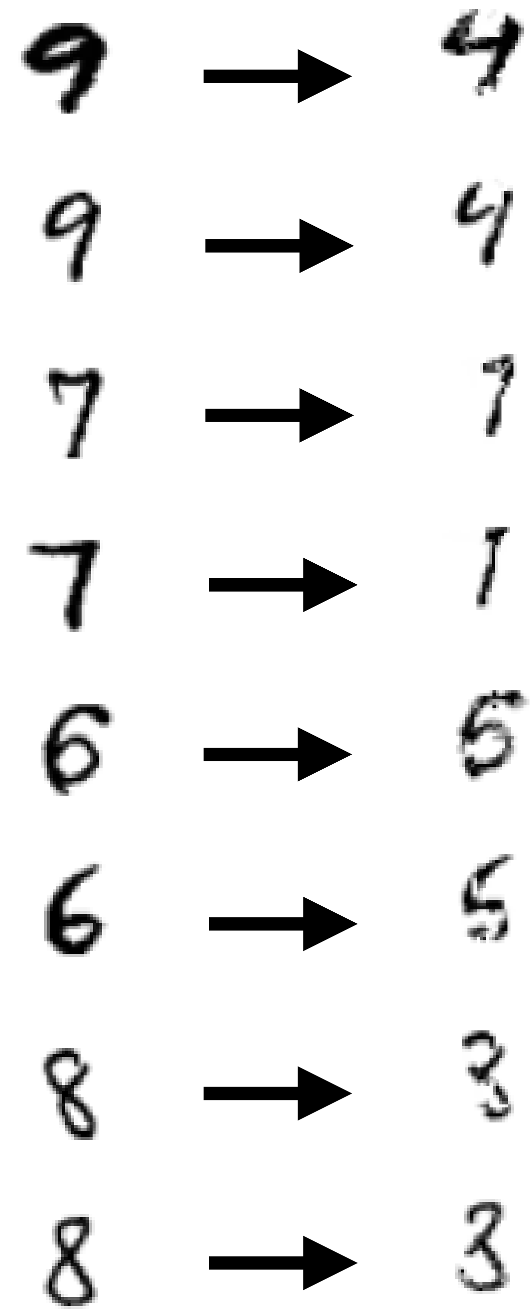


Adversarial Examples

$$\mathbb{P}^\star \in \operatorname*{argmax}_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x^\star, \xi)\right]$$

$$\mathbb{P}^\star \in \operatorname*{argmax}_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \inf_{x\in\mathcal{X}} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x, \xi)\right]$$

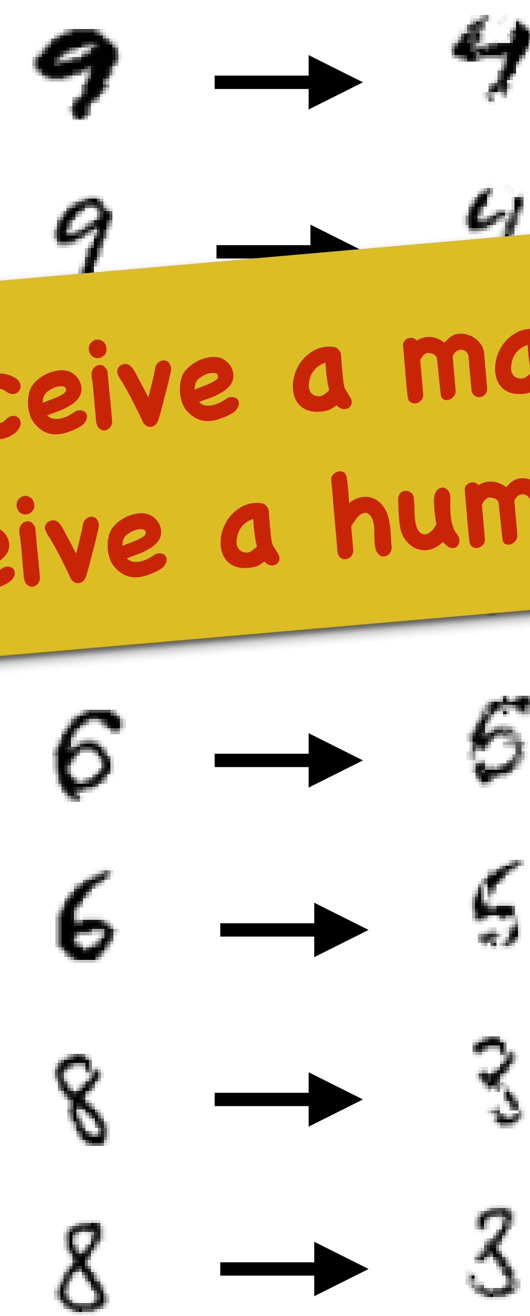




Adversarial Examples

$$\mathbb{P}^\star \in \operatorname*{argmax}_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x^\star,\xi)\right]$$

$$\mathbb{P}^\star \in \operatorname*{argmax}_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \inf_{x\in\mathcal{X}} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right]$$



Best response to $x^\star$ can deceive a machine.
Nash strategy can deceive a human!

From Distributions to Moments

Disadvantages of the Empirical Distribution

Limited scalability: $$\underbrace{\inf_{x\in\mathcal{X}} \sup_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right]}$$

$\cong$ convex program of size $\mathcal{O}(N)$

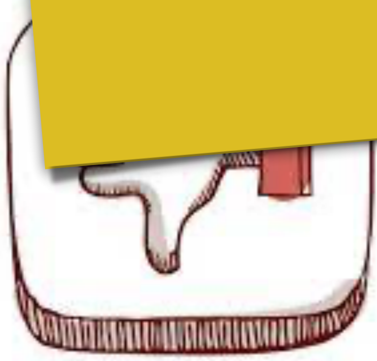
Slow convergence to true distribution

Disadvantages of Working with Distributions

Limited scalability: $\underbrace{\inf_{x \in \mathcal{X}} \sup_{\mathbb{P} \in \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)]}$

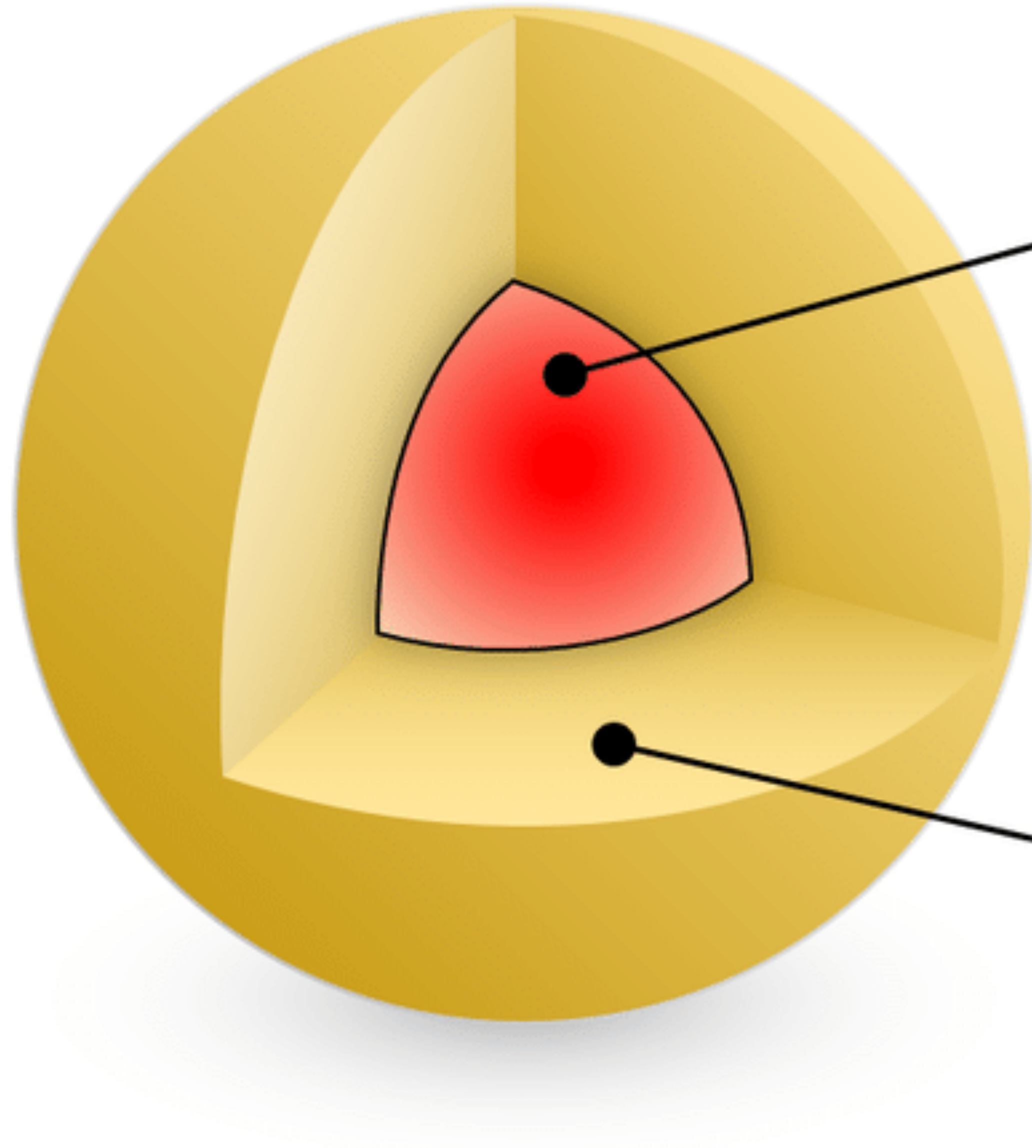
$\simeq$ convex

Focus on 1st and 2nd moments!



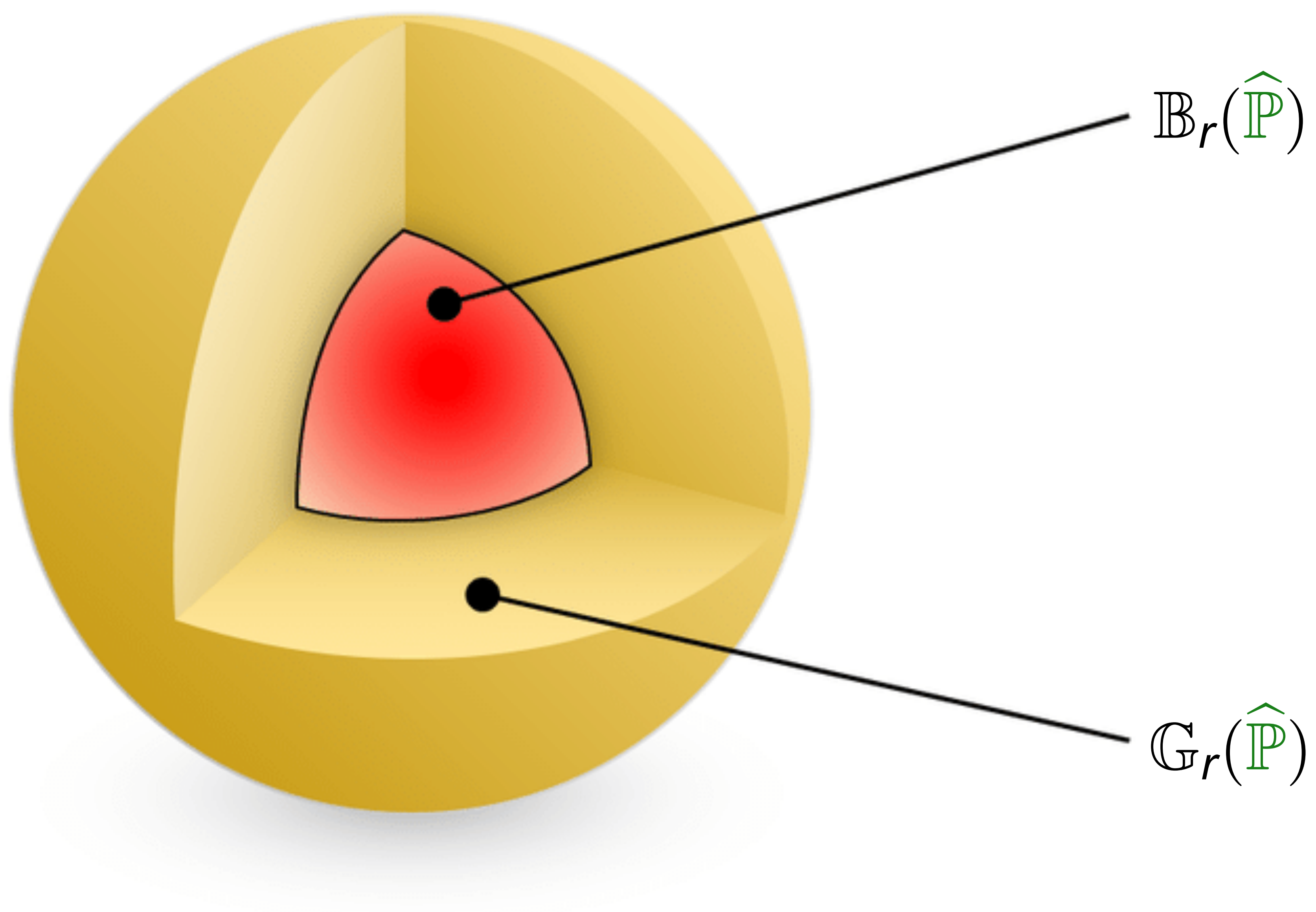
Statistical guarantees difficult to obtain.

Gelbrich Ambiguity Set



$$\mathbb{G}_r(\widehat{\mathbb{P}}) = \left\{ \mathbb{P} \sim (\mu, \Sigma) \;\middle|\; \|\mu - \hat{\mu}\|_2^2 + \mathrm{tr}\left[\Sigma + \hat{\Sigma} - 2\left(\Sigma^{\frac{1}{2}} \hat{\Sigma} \Sigma^{\frac{1}{2}}\right)^{\frac{1}{2}}\right] \leq r^2 \right\}$$

Gelbrich Ambiguity Set

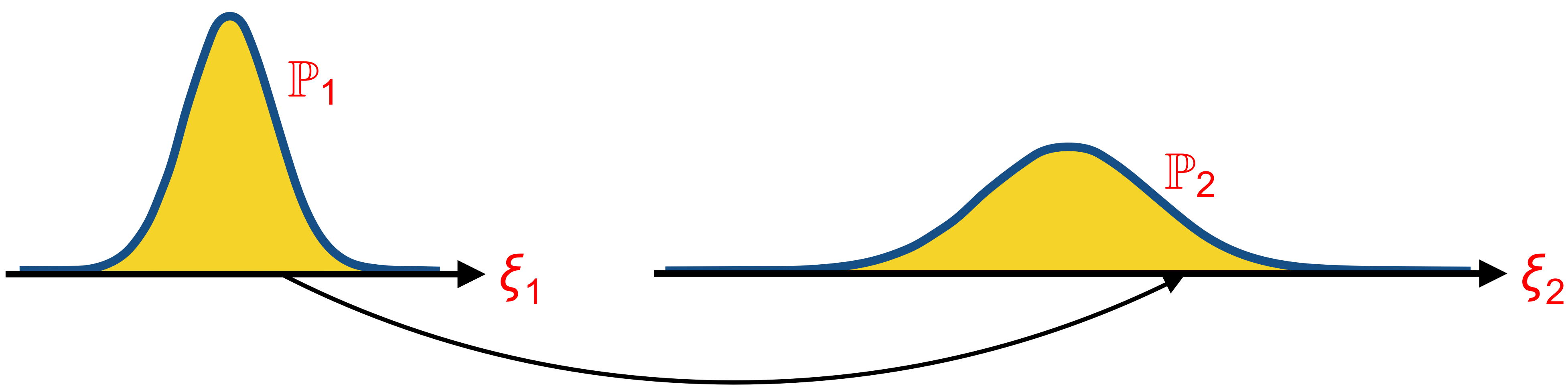


$$\mathbb{G}_r(\widehat{\mathbb{P}}) = \left\{ \mathbb{P} \sim (\mu, \Sigma) \;\middle|\; \underbrace{\|\mu - \hat{\mu}\|_2^2 + \mathrm{tr}\left[\Sigma + \hat{\Sigma} - 2\left(\Sigma^{\frac{1}{2}} \hat{\Sigma} \Sigma^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]}_{\leq W_2^2(\mathbb{P}, \widehat{\mathbb{P}})} \leq r^2 \right\}$$

Structural Information

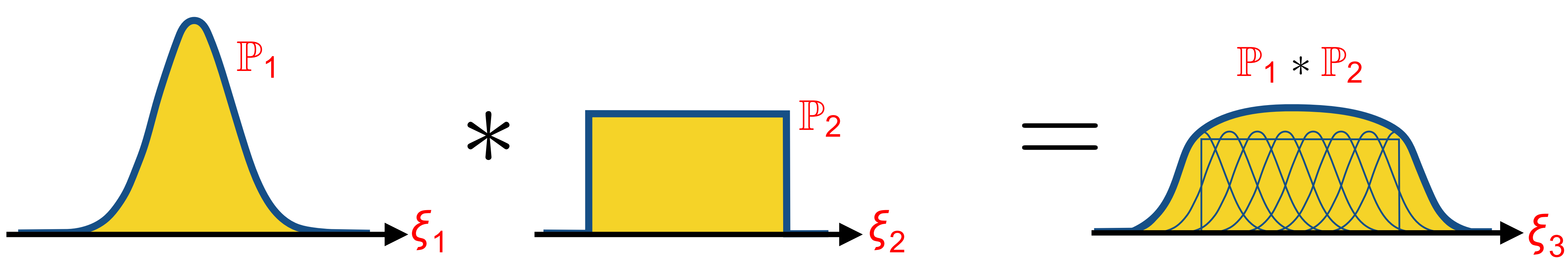
Stable ambiguity set $\mathcal{S}$: Closed under

- affine pushforwards



$\xi_2 = A\xi_1 + b$

- convolutions



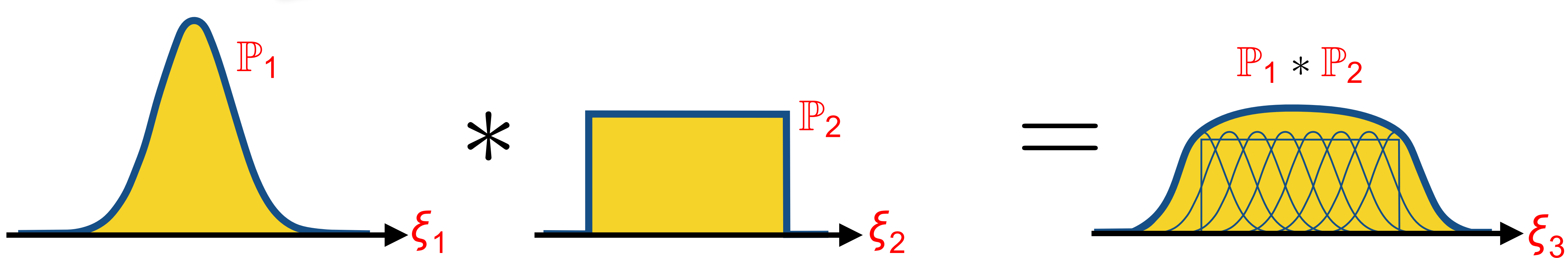
Structural Information

Stable ambiguity set $\mathcal{S}$: Closed under

- affine pushforwards
- convolutio

Examples: All distributions that are…

- symmetric
- symmetric & linear unimodal
- log-concave
- Gaussian



Mean-Covariance Robust Portfolios

Theorem:[1] If $\mathcal{S}$ is stable and $\mathcal{R}_{\xi\sim\mathbb{P}}$ is translation-invariant, scale-invariant and law-invariant, then

$$\sup_{\mathbb{P}\in\mathcal{S}\cap\mathbb{G}_r(\widehat{\mathbb{P}})} \mathcal{R}_{\xi\sim\mathbb{P}}\left[-\xi^\top x\right] = -\hat{\mu}^\top x + \alpha\sqrt{x\hat{\Sigma}x} + r\sqrt{1+\alpha^2}\,\|x\|_2\,.$$

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

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portfolio loss

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

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[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

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nominal standard deviation

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

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regularization term

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

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Standard risk coefficient: $\alpha = \sup_{\substack{\mathbb{P}\in\mathcal{S}\\ \mathbb{P}\sim(\mu,\Sigma)}} \mathcal{R}_{\xi\sim\mathbb{P}}\left[-\frac{(\xi-\mu)^\top x}{\sqrt{x^\top\Sigma x}}\right]$

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

Mean-Covariance Robust Portfolios

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independent of x, μ and Σ

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

Mean-Covariance Robust Portfolios

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Proposition: If $\mathcal{R}_{\xi\sim\mathbb{P}} = \beta\text{-VaR}_{\xi\sim\mathbb{P}}$, then

$$\alpha = \begin{cases} \sqrt{(1-\beta)/\beta} & \mathcal{S} = \{\text{all distributions}\} \\ 1/\sqrt{2\beta} & \mathcal{S} = \{\text{all symmetric distributions}\} \\ 1/(3\sqrt{2\beta}) & \mathcal{S} = \{\text{all symm. lin. unimodal distributions}\} \\ \Phi^{-1}(1-\beta) & \mathcal{S} = \{\text{all Gaussian distributions}\} \end{cases}$$

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

Mean-Covariance Robust Portfolios

Theorem:[1] If $\mathcal{S}$ is stable and $\mathcal{R}_{\xi\sim\mathbb{P}}$ is translation-invariant, scale-invariant and law-invariant, then

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α available analytically if $\mathcal{S} = \{\text{all distributions}\}$ and $\mathcal{R}_{\xi\sim\mathbb{P}}$ is

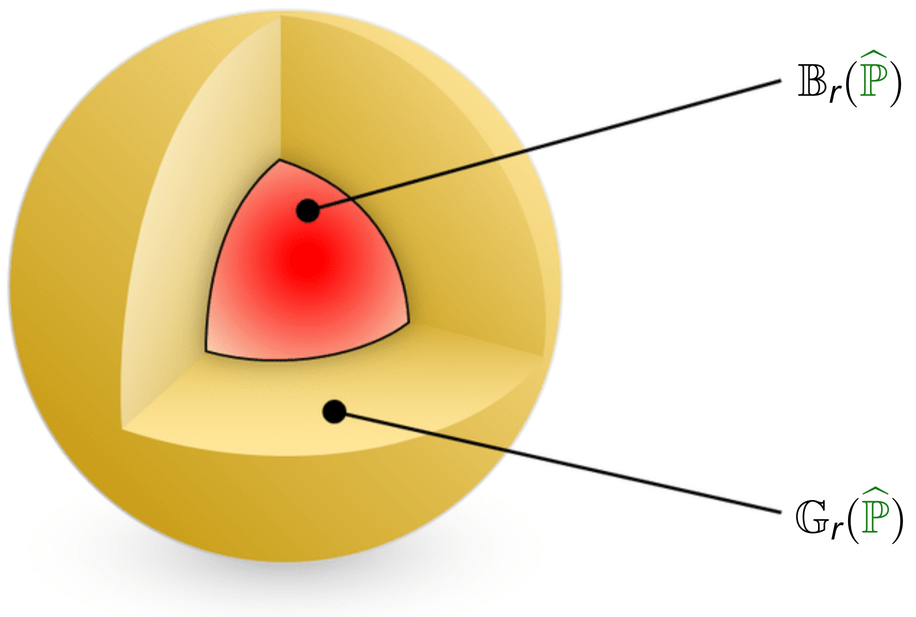
- any **spectral** risk measure;
- any risk measure with a **Kusuoka representation**;
- any **distortion** risk measure.

[1] Nguyen, Shafieezadeh-Abadeh, Filipovic & Kuhn, *Working Paper*, 2022.

Scalable Wasserstein DRO

Conservative approximation:

$$\sup_{\mathbb{P}\in\mathcal{S}\cap\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right] \leq \sup_{\mathbb{P}\in\mathcal{S}\cap\mathbb{G}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right]$$



Scalable Wasserstein DRO

Conservative approximation:

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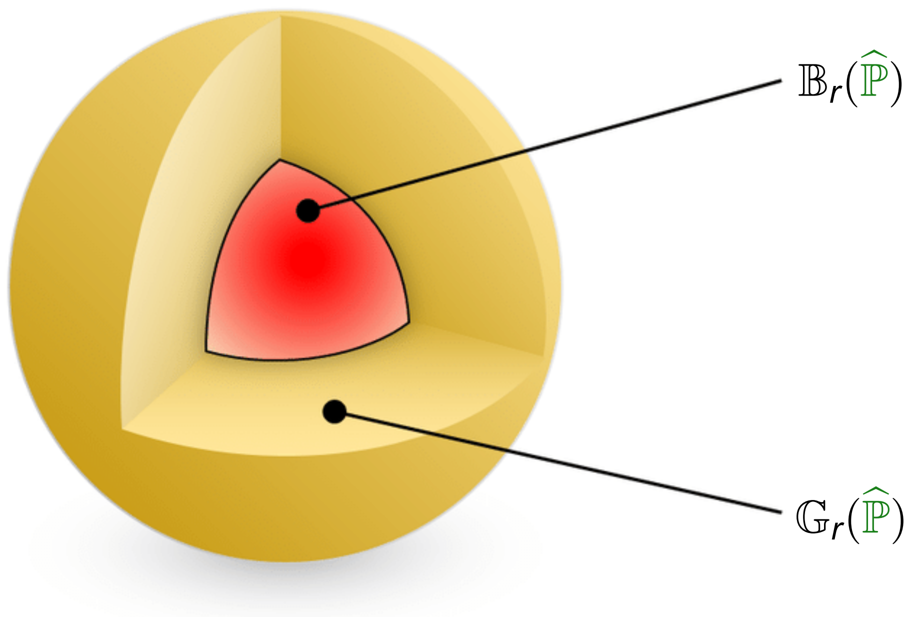
$$\sup_{\mathbb{P}\in\mathcal{S}\cap\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right] \leq \underbrace{\sup_{\mathbb{P}\in\mathcal{S}\cap\mathbb{G}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi\sim\mathbb{P}}\left[\ell(x,\xi)\right]}_{\text{independent of sample size}}$$



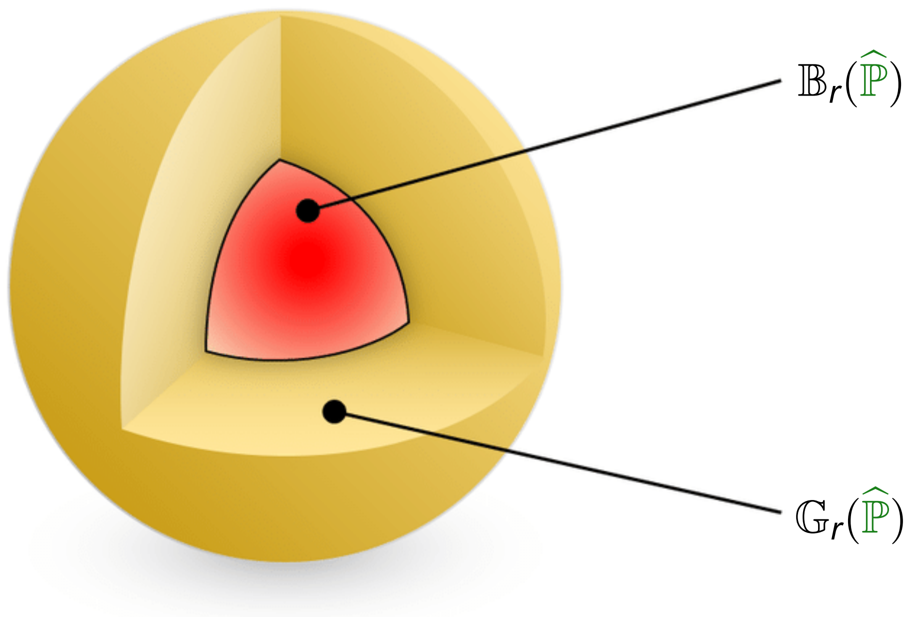
Scalable Wasserstein DRO

Conservative approximation:

$$\sup_{\mathbb{P} \in \mathcal{S} \cap \mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)] \leq \sup_{\mathbb{P} \in \mathcal{S} \cap \mathbb{G}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\xi \sim \mathbb{P}} [\ell(x, \xi)]$$

exact if…

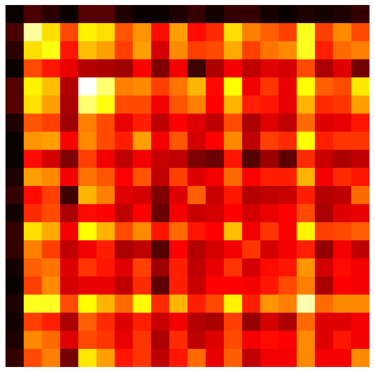
- $\ell(x, \xi)$ is quadratic in ξ or
- $\mathcal{S} = \{$all Gaussian distributions$\}$



Wasserstein DRO with Quadratic Loss

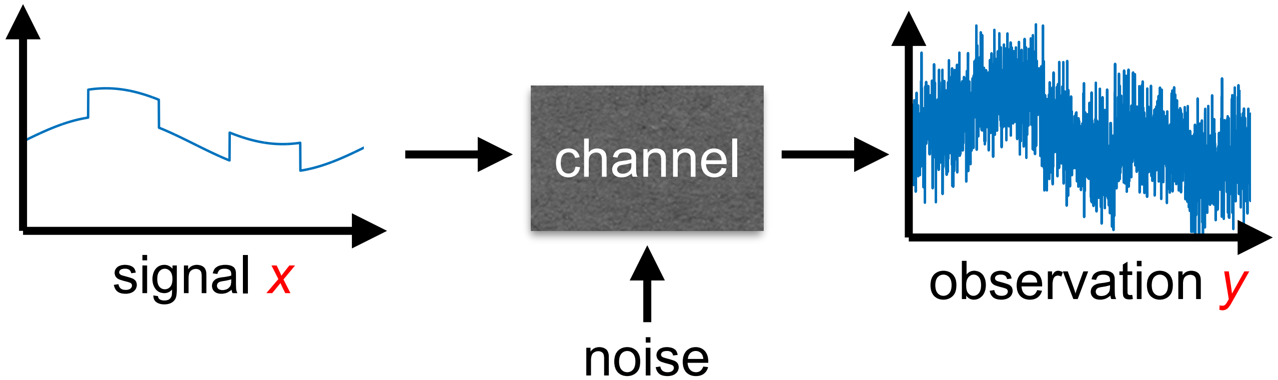
Inverse covariance estimation:[1)]

$$\inf_{X \succ 0} \; -\log\det X + \sup_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}\left[\xi^\top X \xi\right]$$

$X =$ 

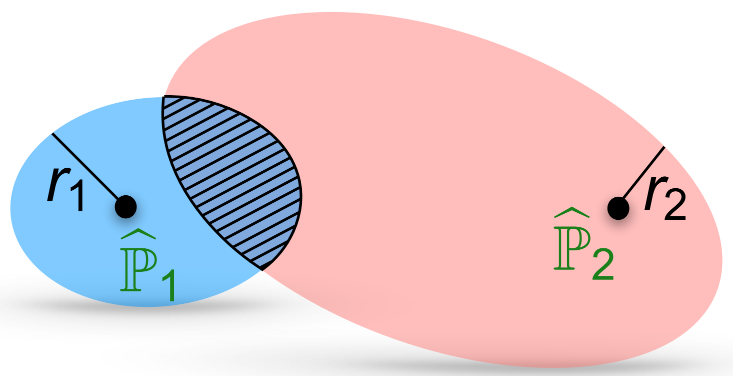
Signal processing & Kalman filtering:[2)]

$$\inf_{\psi(\cdot)} \sup_{\mathbb{P}\in\mathbb{B}_r(\widehat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}\left[\|x-\psi(y)\|_2^2\right]$$



Domain adaptation:[3)]

$$\inf_{\beta} \sup_{\mathbb{P}\in\cap_k\mathbb{B}_{r_k}(\widehat{\mathbb{P}}_k)} \mathbb{E}_{\mathbb{P}}\left[(\beta^\top x - y)^2\right]$$



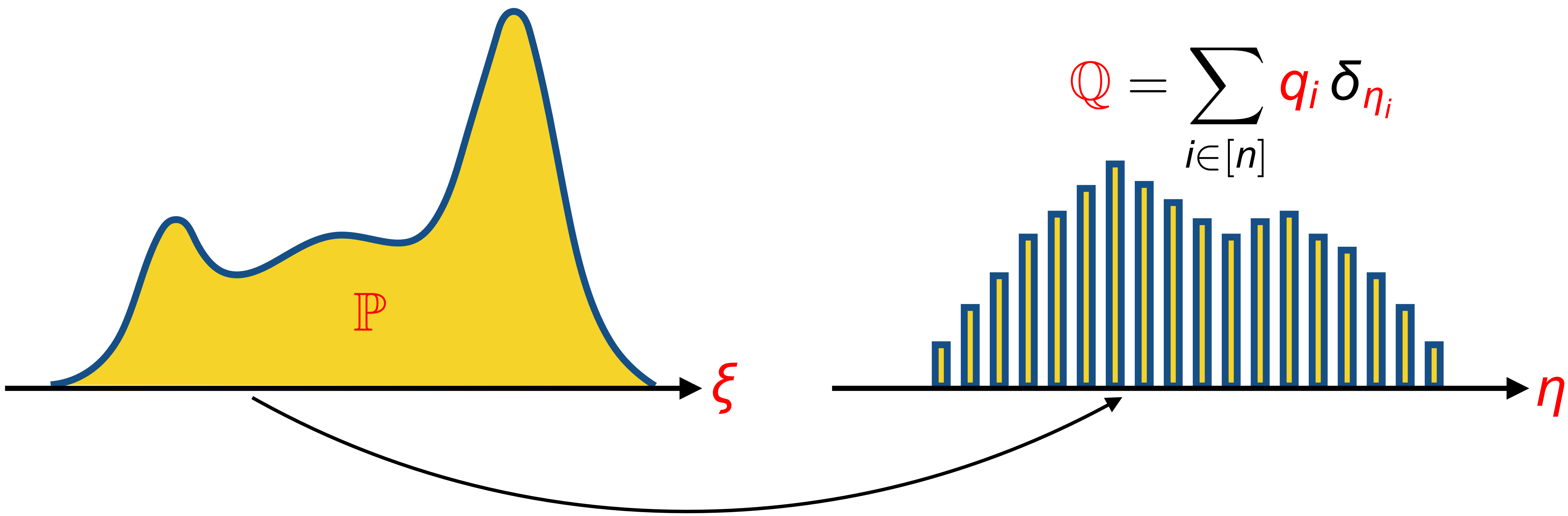
1) Nguyen, Kuhn & Mohajerin Esfahani, *Oper. Res.*, 2022.
2) Nguyen, Shafieezadeh-Abadeh, Kuhn & Mohajerin Esfahani, *Math. Oper. Res.*, 2023.
3) Taskesen, Yue, Blanchet, Kuhn & Nguyen, *ICML*, 2021.

Ask not what OT can do for DRO —
ask what DRO can do for OT.

Semi-Discrete OT

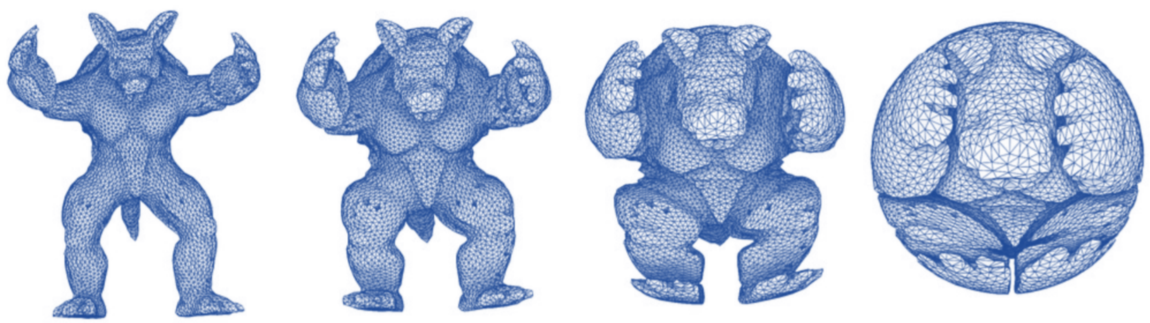
$$\min_{\pi \in \Pi(\mathbb{P}, \mathbb{Q})} \mathbb{E}_{(\xi, \eta) \sim \pi} \left[c(\xi, \eta) \right]$$

$$\mathbb{Q} = \sum_{i \in [n]} q_i \, \delta_{\eta_i}$$

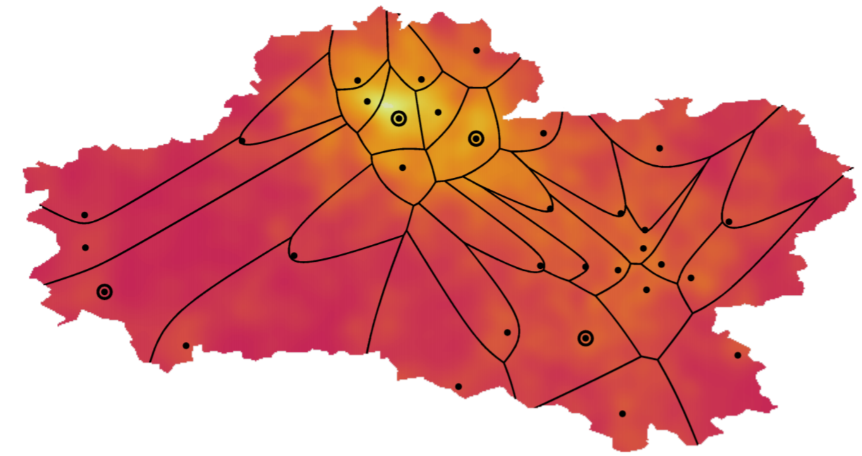


Applications of Semi-Discrete OT

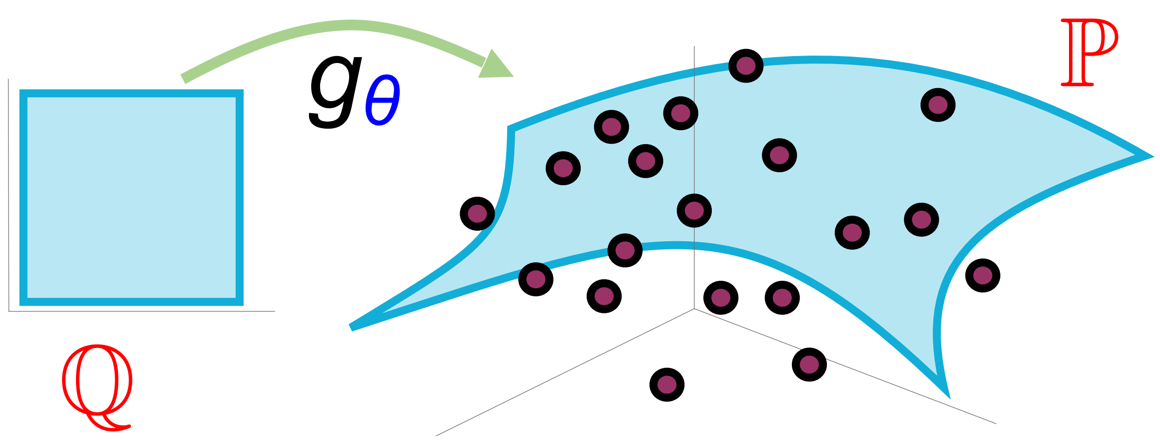
3D morphing[1)]



Resource allocation[2)]

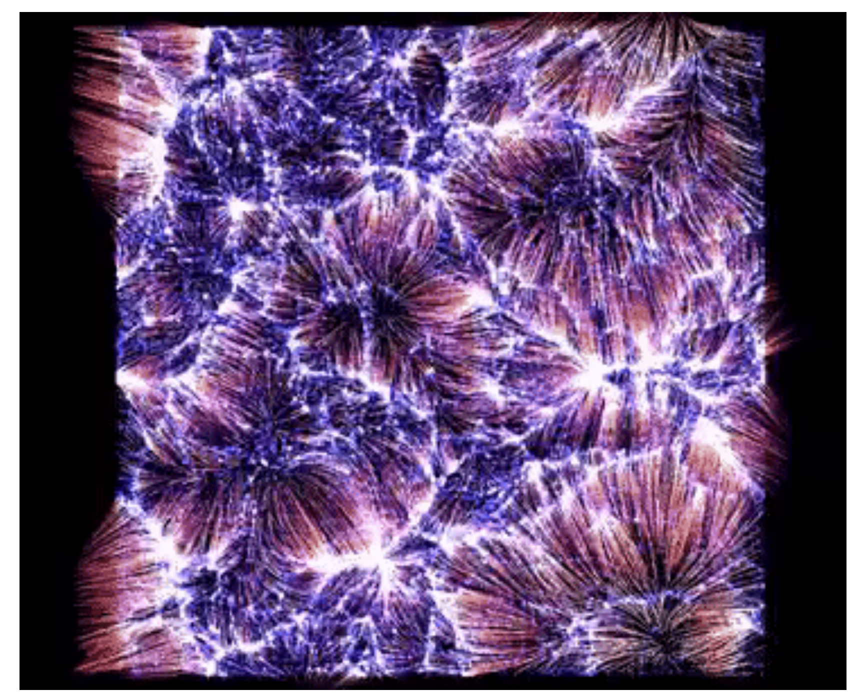


Generative models[3)]



$$\min_{\theta} W(\mathbb{P}, g_{\theta\#}\mathbb{Q})$$

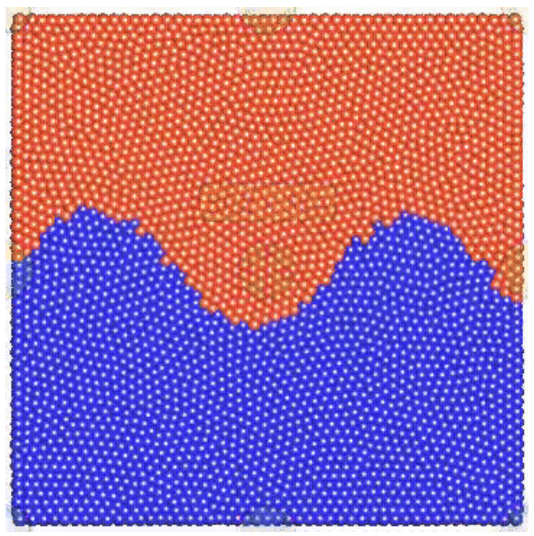
Reconstruction of the early universe[4)]



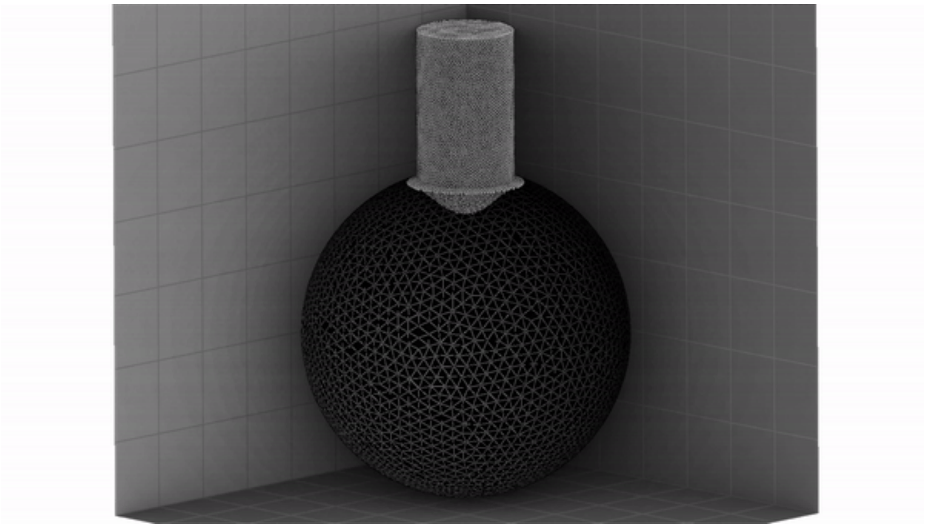
[1)] Lévy, *ESAIM: M2AN,* 2014.
[2)] Hartmann & Schuhmacher, *Math. Meth. Oper. Res.,* 2020.
[3)] Arjovsky, Chintala & Bottou, *ICML,* 2017.
[4)] Lévy, Mohayaee, von Hausegger & Natarajan, *MNRAS*, 2021.

Applications of Semi-Discrete OT

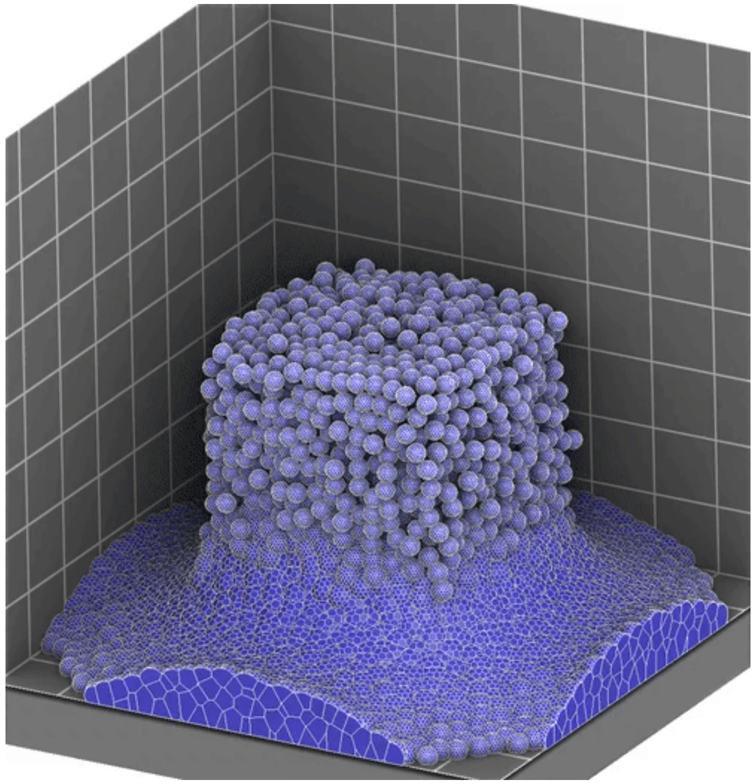
Taylor-Rayleigh instability using the Gallouet-Merigot scheme[1)]



Simulation of an incompressible bi-phasic flow in a bottle[1)]



Free-surface fluid simulation with Gallouet-Merigot scheme[1)]



1) Goes, Wallez & Huang, *ACM Trans. Graph.*, 2015; Gallouet & Merigot, *Found. Comput. Math.*, 2017; Lévy, *arXiv*, 2018.

Duality

primal OT: $$\min_{\pi \in \Pi(\mathbb{P}, \mathbb{Q})} \mathbb{E}_{(\xi, \eta) \sim \pi} \left[c(\xi, \eta) \right]$$

Duality

primal OT: $$\min_{\pi \in \Pi(\mathbb{P}, \mathbb{Q})} \mathbb{E}_{(\xi, \eta) \sim \pi} [c(\xi, \eta)]$$

dual OT: $$\sup_{\phi \in \mathbb{R}^n} \sum_{i \in [n]} q_i \phi_i - \mathbb{E}_{\xi \sim \mathbb{P}} [\psi_c(\phi, \xi)]$$

Duality

<u>primal OT</u>:

$$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right]$$

<u>dual OT</u>:

$$\sup_{\phi \in \mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\boxed{\psi_c(\phi,\xi)}\right]$$

$$\psi_c(\phi,\xi) = \max_{i\in[n]} \phi_i - c(\xi,\eta_i)$$

Relation to Discrete Choice Theory

primal OT: $$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right]$$

dual OT: $$\sup_{\phi \in \mathbb{R}^n} \sum_{i \in [n]} q_i \phi_i - \mathbb{E}_{\xi \sim \mathbb{P}} \boxed{\psi_c(\phi,\xi)}$$

Luis Thurstone
1927

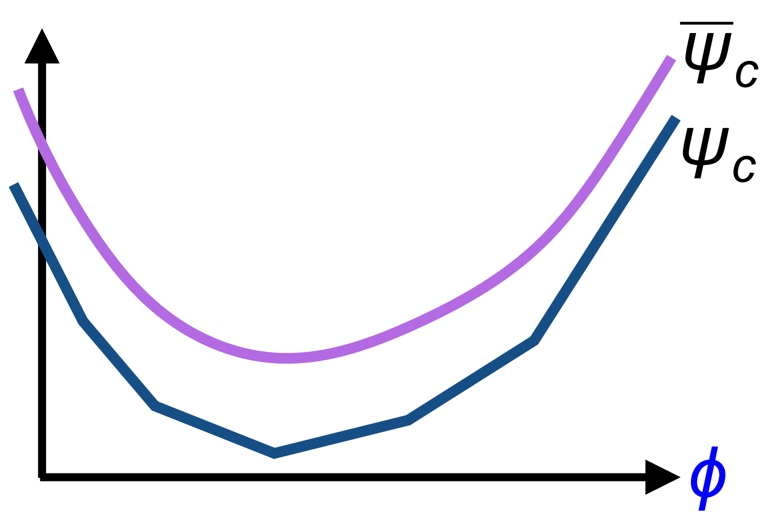
Daniel McFadden
1978

$$\underbrace{\psi_c(\phi,\xi) = \max_{i \in [n]} \phi_i - c(\xi,\eta_i)}_{\text{deterministic discrete choice model}}$$

Relation to Discrete Choice Theory

primal OT:
$$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right]$$

dual OT:
$$\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\psi_c(\phi,\xi)\right]$$



$$\underbrace{\overline{\psi}_c(\phi,\xi) = \sup_{\theta\in\Theta} \mathbb{E}_{z\sim\theta}\left[\max_{i\in[n]} \phi_i - c(\xi,\eta_i) + z_i\right]}_{\text{distributionally robust discrete choice problem}^{1)}}$$

distributionally robust
discrete choice problem[1)]

1) Natarajan, Song & Teo, *Manag. Sci.*, 2009.

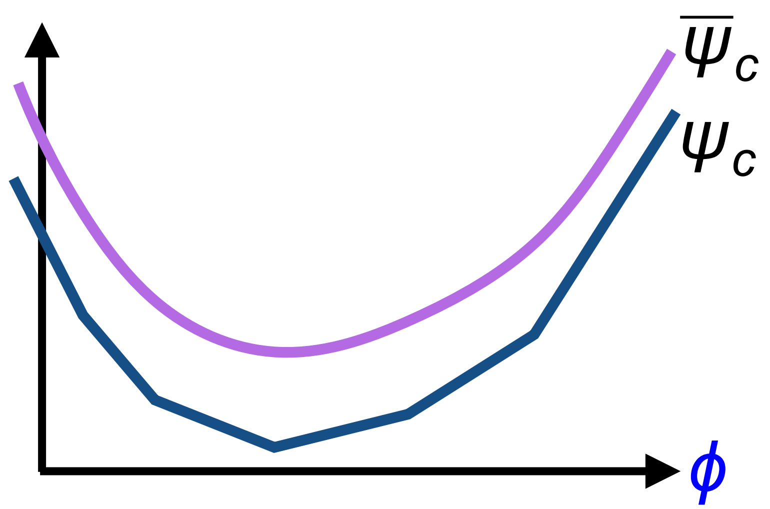
Relation to Discrete Choice Theory

primal OT:

$$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right]$$

dual OT:

$$\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\psi_c(\phi,\xi)\right]$$



$$\overline{\psi}_c(\phi,\xi) = \sup_{\theta\in\boxed{\Theta}} \mathbb{E}_{z\sim\theta}\left[\max_{i\in[n]} \phi_i - c(\xi,\eta_i) + z_i\right]$$

Frechet ambiguity set[1)]

$$\Theta = \{\theta \mid \theta(z_i \le s) \equiv F_i(s)\ \forall i \in [n]\}$$

---

1) Natarajan, Song & Teo, *Manag. Sci.*, 2009.

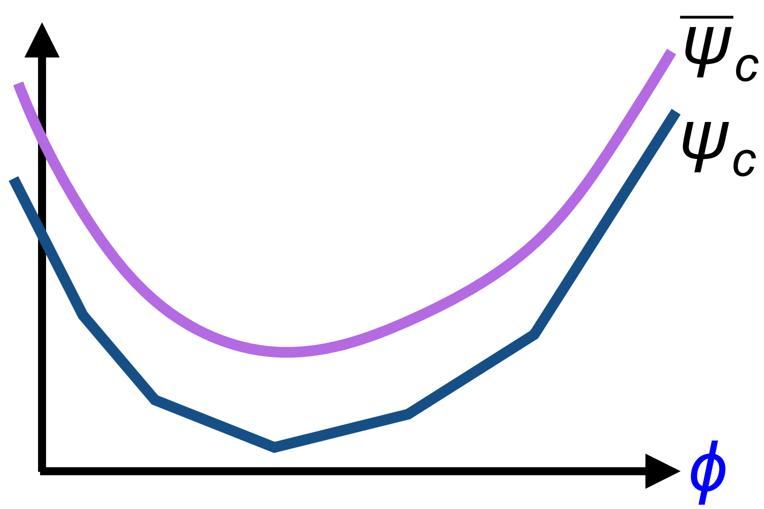
Smoothing the Dual Objective

primal OT:

$$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \; \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right]$$

dual OT:

$$\sup_{\phi \in \mathbb{R}^n} \; \sum_{i\in[n]} q_i \phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\psi_c(\phi,\xi)\right]$$



$$\overline{\psi}_c(\phi,\xi) = \sup_{\theta\in\Theta} \; \mathbb{E}_{z\sim\theta}\left[\max_{i\in[n]} \phi_i - c(\xi,\eta_i) + z_i\right]$$

Smoothing the Dual Objective

primal OT:

$$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right]$$

smooth dual OT:

$$\sup_{\phi \in \mathbb{R}^n} \sum_{i\in[n]} q_i \phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\overline{\psi}_c(\phi,\xi)\right]$$

Regularizing the Primal Objective

<u>regularized primal OT</u>:

$$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right] + D_f(\pi \| \mathbb{P}\otimes\mathbb{Q})$$

<u>smooth dual OT</u>:

$$\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\overline{\psi}_c(\phi,\xi)\right]$$

Duality Revisited

regularized primal OT: $$\min_{\pi\in\Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right] + D_f(\pi\|\mathbb{P}\otimes\mathbb{Q})$$

smooth dual OT: $$\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\overline{\psi}_c(\phi,\xi)\right]$$

Theorem:[1)] Smooth dual OT is equivalent to regularized primal OT if $F_i(s) = \mathrm{proj}_{[0,1]}(1 - q_i F(-s))$ for all $i \in [n]$ and $f(s) = \int_0^s F^{-1}(t)\,\mathrm{d}t$.

1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Duality Revisited

<u>regularized primal OT</u>: $$\min_{\pi \in \Pi(\mathbb{P},\mathbb{Q})} \mathbb{E}_{(\xi,\eta)\sim\pi}\left[c(\xi,\eta)\right] + D_f(\pi \| \mathbb{P}\otimes\mathbb{Q})$$

<u>smooth dual OT</u>: $$\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\overline{\psi}_c(\phi,\xi)\right]$$

Theorem:[1] Smooth dual OT is equivalent to regularized primal OT if $F_i(s) = \mathrm{proj}_{[0,1]}(1 - q_i \boxed{F}(-s))$ for all $i \in [n]$ and $f(s) = \int_0^s \boxed{F}^{-1}(t)\,\mathrm{d}t$.

marginal generator

1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Unification of Regularization Schemes

F	Regularizer
Exponential	KL divergence[1)]
Uniform	χ^2-divergence[2)]
Pareto	Tsallis divergence[3)]
Hyperbolic cosine	Hyperbolic divergence
t-distribution	Chebychev

[1)] Cuturi, *NeurIPS*, 2013; Genevay, Cuturi, Peyré & Bach, *NeurIPS*, 2016.
[2)] Blondel, Seguy & Rolet, *AISTATS*, 2018; Seguy, Damodaran, Flamary *et. al.*, *ICLR*, 2018.
[3)] Muzellec, Nock, Patrini & Nielsen, *AAAI*, 2017.

Inexact Averaged SGD[1)]

smooth dual OT: $$\sup_{\phi \in \mathbb{R}^n} \sum_{i \in [n]} q_i \phi_i - \mathbb{E}_{\xi \sim \mathbb{P}} \left[\overline{\psi}_c(\phi, \xi) \right]$$

1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Inexact Averaged SGD[1)]

smooth dual OT: $$\sup_{\phi \in \mathbb{R}^n} \sum_{i \in [n]} q_i \phi_i - \mathbb{E}_{\xi \sim \mathbb{P}} \boxed{\overline{\psi}_c(\phi, \xi)}$$

unbiased stochastic gradient ⇔ optimal choice probabilities

1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Inexact Averaged SGD[1)]

smooth dual OT: $\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}} \boxed{\overline{\psi}_c(\phi, \xi)}$

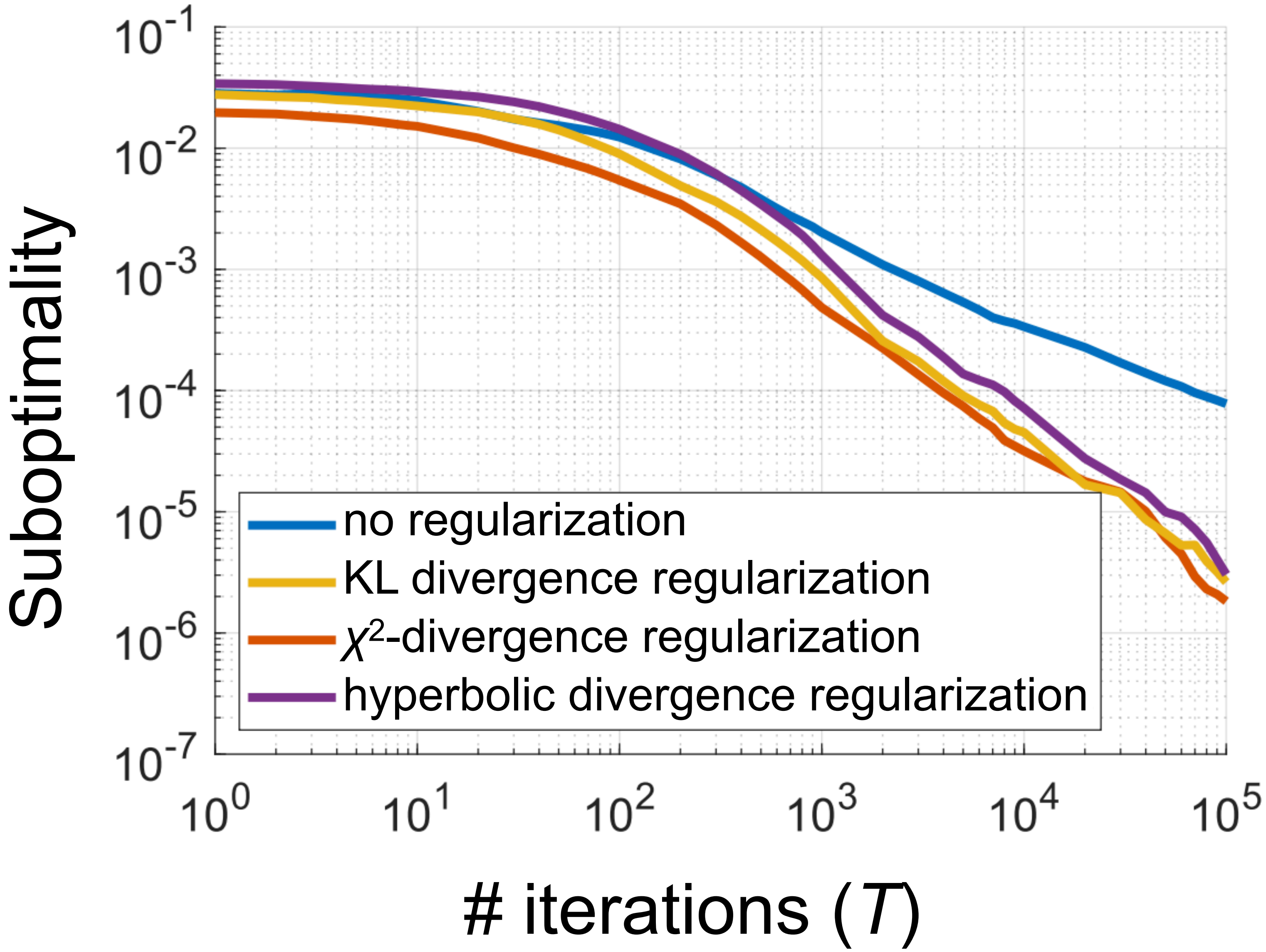
unbiased stochastic gradient ⇔ optimal choice probabilities

⇒ benefit from rich literature on discrete choice theory!

1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Inexact Averaged SGD[1)]

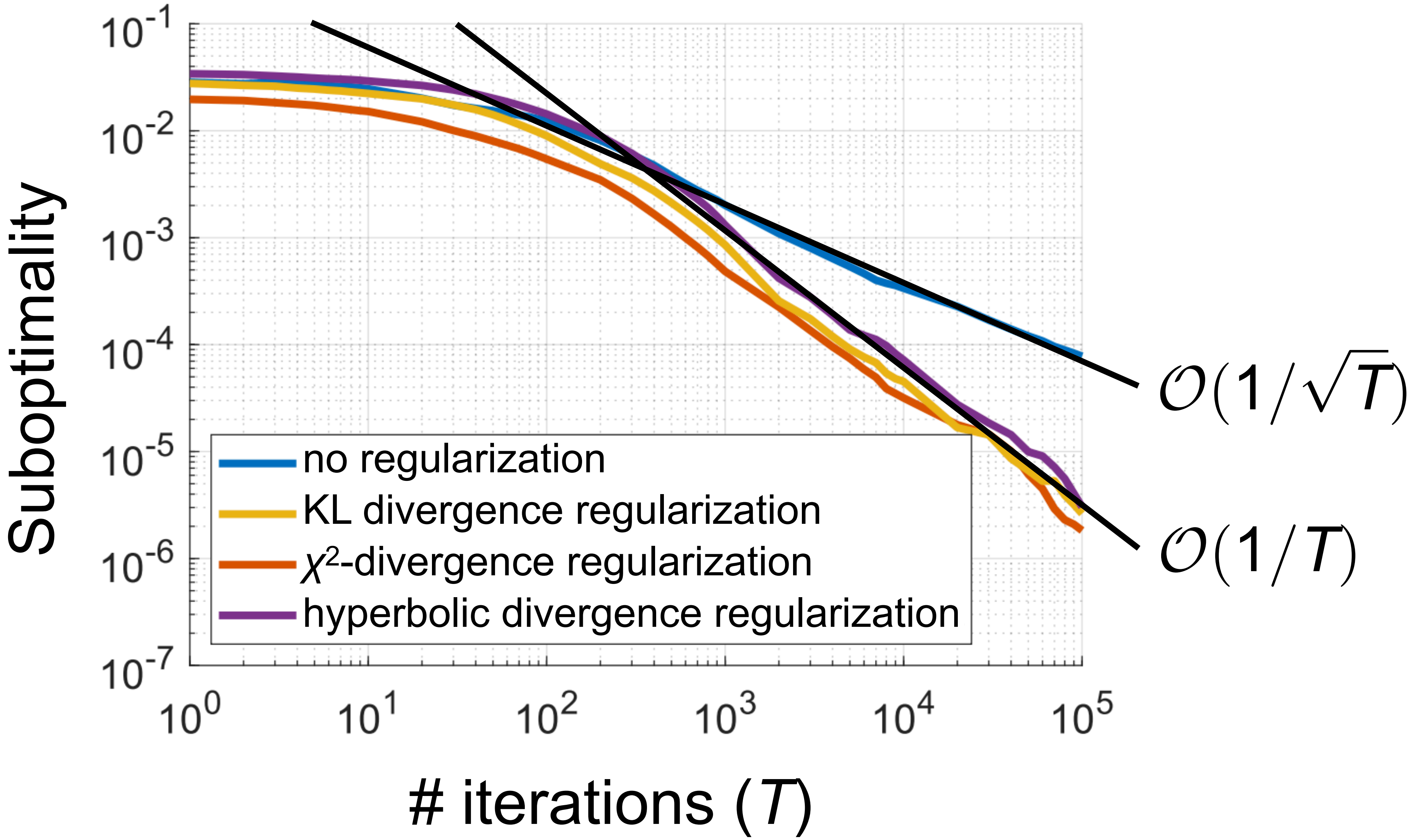
smooth dual OT: $$\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\overline{\psi}_c(\phi,\xi)\right]$$



1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Inexact Averaged SGD[1)]

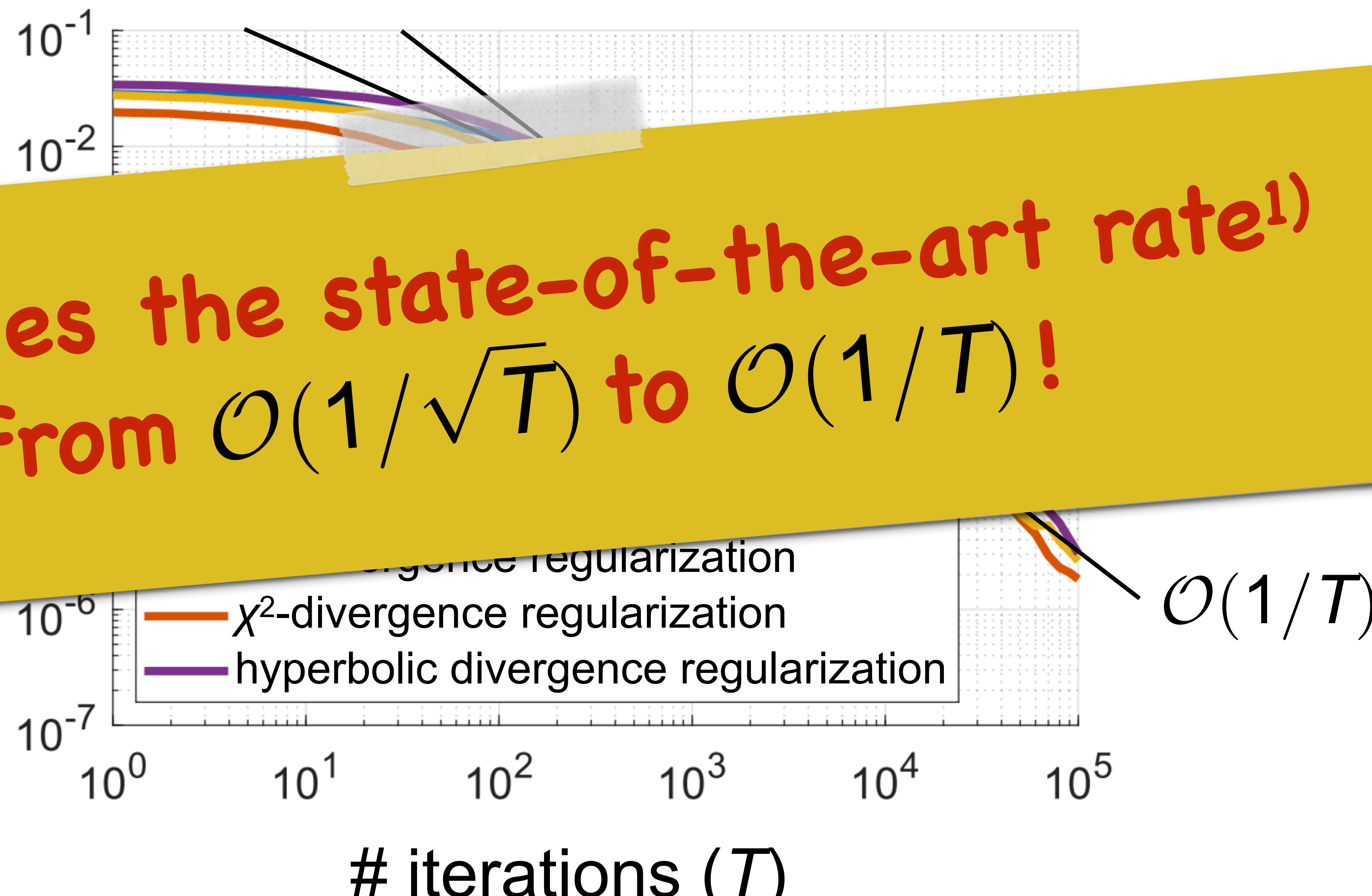
<u>smooth dual OT</u>: $$\sup_{\phi \in \mathbb{R}^n} \sum_{i \in [n]} q_i \phi_i - \mathbb{E}_{\xi \sim \mathbb{P}} \left[\overline{\psi}_c(\phi, \xi) \right]$$



1) Taskesen, Shafieezadeh-Abadeh & Kuhn, *Math. Program.*, 2023.

Inexact Averaged SGD

smooth dual OT: $$\sup_{\phi\in\mathbb{R}^n} \sum_{i\in[n]} q_i\phi_i - \mathbb{E}_{\xi\sim\mathbb{P}}\left[\overline{\psi}_c(\phi,\xi)\right]$$



Improves the state-of-the-art rate[1)] from $\mathcal{O}(1/\sqrt{T})$ to $\mathcal{O}(1/T)$!

1) Genevay, Cuturi, Peyré & Bach, *NeurIPS*, 2016.

Summary

- **Wasserstein DRO**
  - offers **statistical guarantees**
  - **tractable** in generic settings
  - provides probabilistic justification of **regularization**
  - leads to **adversarial examples** that can deceive humans

- **Gelbrich DRO**
  - highly **scalable approximation** using 1^{st} and 2^{nd} moments
  - exact for **quadratic** loss functions or **Gaussian** distributions

- **Semi-Discrete OT**
  - **#P-hard** but amenable to **fast SGD algo.** by adding noise
  - **dual smoothing** equivalent to **primal regularization**

This Talk is Based on…

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[2] P. Mohajerin Esfahani & D. Kuhn. **Data-Driven Distributionally Robust Optimization using the Wasserstein Metric: Performance Guarantees and Tractable Reformulations**. *Mathematical Programming* 171(1–2), 115–166, 2018.

[3] V. Nguyen, D. Kuhn & P. Mohajerin Esfahani. **Distributionally Robust Inverse Covariance Estimation: The Wasserstein Shrinkage Estimator**. *Operations Research* 70(1), 490–515, 2018.

[4] V. Nguyen, S. Shafieezadeh-Abadeh, D. Filipovic & D. Kuhn. **Mean-Covariance Robust Risk Measurement**. Working paper, 2021.

[5] V. Nguyen, S. Shafieezadeh-Abadeh, D. Kuhn & P. Mohajerin Esfahani. **Bridging Bayesian and Minimax Mean Square Error Estimation via Wasserstein Distributionally Robust Optimization**. *Mathematics of Operations Research* 48 (1), 1-37, 2023.

[6] S. Shafieezadeh-Abadeh, L. Aolaritei, F. Dörfler & D. Kuhn. **Optimal Transport for Distributionally Robust Optimization: Nash Equilibria, Regularization, and Computation**. Working paper, 2023.

[7] S. Shafieezadeh-Abadeh, P. Mohajerin Esfahani & D. Kuhn. **Distributionally Robust Logistic Regression**. *Neural Information Processing Systems*, 2015.

[8] S. Shafieezadeh-Abadeh, P. Mohajerin Esfahani & D. Kuhn. **Regularization via Mass Transportation**. *Journal of Machine Learning Research* 20(103), 1–68, 2019.

[9] S. Shafieezadeh-Abadeh, P. Mohajerin Esfahani, V. Nguyen & D. Kuhn. **Wasserstein Distributionally Robust Kalman Filtering**. *Neural Information Processing Systems*, 2018.

[10] B. Taşkesen, M.-C. Yue, J. Blanchet, D. Kuhn & V. Nguyen. **Sequential Domain Adaptation by Synthesizing Distributionally Robust Experts**. *International Conference on Machine Learning*, 2021.

[11] B. Taşkesen, S. Shafieezadeh-Abadeh & D. Kuhn. **Semi-Discrete Optimal Transport: Hardness, Regularization and Numerical Solution**. *Mathematical Programming* 199 (1-2), 1033-1106, 2023.

[12] J. (T.) Zhen, D. Kuhn & W. Wiesemann. **A Unified Theory of Robust and Distributionally Robust Optimization via the Primal-Worst-Equals-Dual-Best Principle**. *Operations Research*, 2023.