SIPTA Seminar 20 March 2026

What is a good notion of independence for choice functions?

Arthur Van Camp

Eindhoven University of Technology

Independence

Consider two random variables X and Y, taking values in finite possibility spaces, with joint probability mass function (pmf) $p_{X,Y}$. Then X and Y are independent if

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \quad \text{for all } x \text{ and } y. \qquad \text{(factorisation)}$$

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$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = p_X(x) \quad \text{and} \quad p_{Y|X}(y \mid x) = p_Y(y) \quad \text{for all } x \text{ and } y.$$

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There are two other, plausible, interpretations of independence:

2. There is an independent representation.

3. You should not spend resources in order to use the observed value of Y to choose between decisions that only depend on X, and vice versa.

Overview

Part 1: Sets of desirable gambles and choice functions

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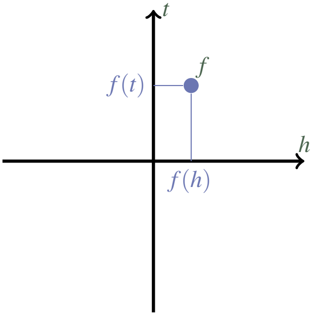

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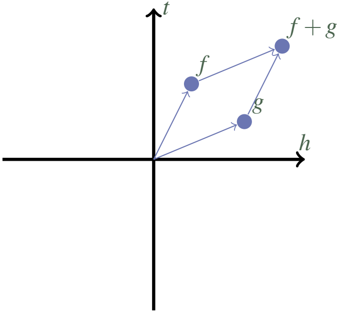

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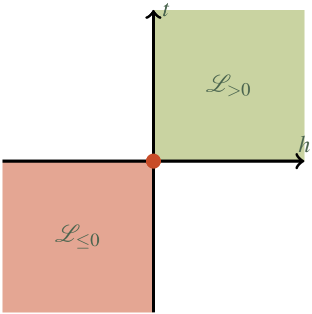

We say that $f < g$ if
$f(x) \leq g(x)$ for all x in $\mathscr{X}$ and $f(x) < g(x)$ for some x in $\mathscr{X}$.

SETS OF DESIRABLE GAMBLES

Sets of desirable gambles

If you have a pmf p, then you have a preference order $\prec_p$ given by

$$f \prec_p g \Leftrightarrow E_p(f) < E_p(g) \quad \text{for all gambles } f \text{ and } g.$$

$\prec_p$ is a strict weak order that is moreover compatible with vector operations: $f \prec_p g \Leftrightarrow \lambda f + h \prec_p \lambda g + h$ for all gambles f, g and h and real $\lambda > 0$.

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$$f \prec g \Leftrightarrow 0 \prec g - f \Leftrightarrow g - f \in D.$$

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Sets of desirable gambles: coherence

$\mathscr{L}$: linear space of gambles

$D \subseteq \mathscr{L}$: set of desirable gambles

Your set of desirable gambles D is the set of gambles you prefer to the status quo:
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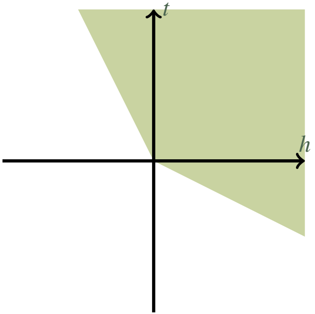

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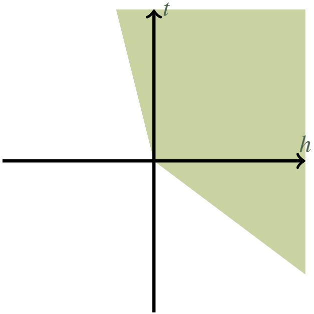

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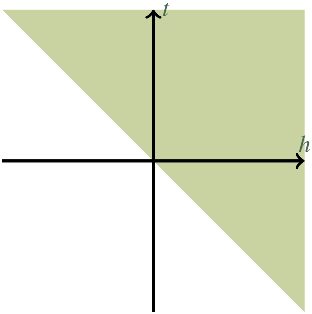

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Given a pmf p, the set $D_p := \{f \in \mathscr{L} : E_p(f) > 0\} \cup \mathscr{L}_{>0}$ is coherent.

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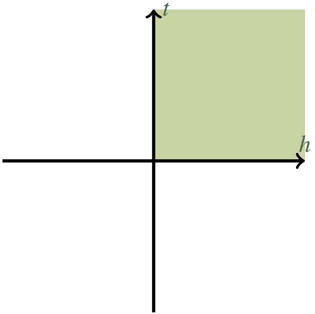

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There a unique smallest coherent D: the vacuous set $D_v = \mathscr{L}_{>0}$.

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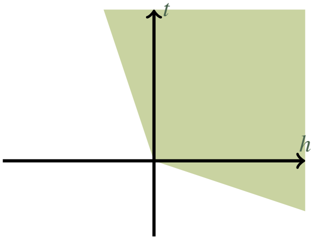

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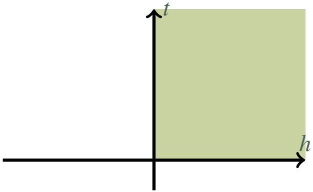

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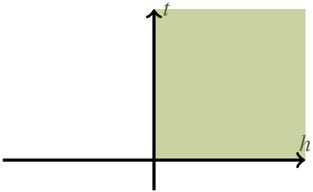

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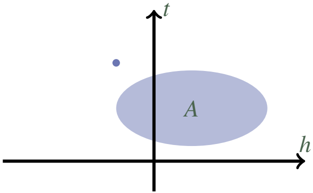

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Then by coherence gambles outside A may be desirable: by D$_3$ and D$_4$ any gamble in

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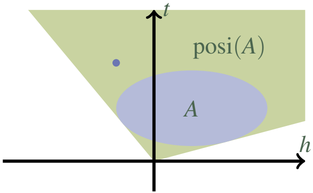
is desirable as well.



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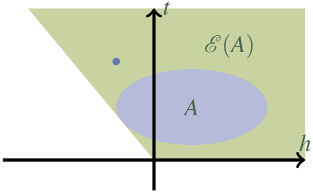
Theorem (Natural extension)

An assessment $A \subseteq \mathscr{L}$ has a coherent extension – is consistent – if and only if $0 \notin \text{posi}(A \cup \mathscr{L}_{>0})$.

If A is consistent, then its natural extension $\mathscr{E}(A)$ – its smallest coherent superset – is $\mathscr{E}(A) = \text{posi}(A \cup \mathscr{L}_{>0})$.



Sets of desirable gambles: information

Find more information in the work of, among others:

Teddy Seidenfeld

Peter Walley

Peter Williams

Gert de Cooman

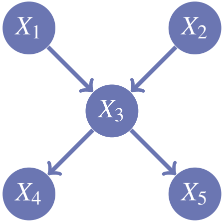
Erik Quaeghebeur

Jasper De Bock

Enrique Miranda

Sets of desirable gambles: information

Interesting application: credal networks under epistemic irrelevance.



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The local Markov condition becomes: For each node, its non-parents non-descendants are irrelevant to it, given its parents.

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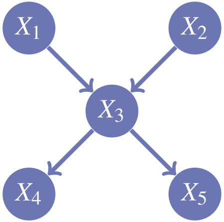


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Conditional on X_3, the variables X_4 and X_5 are epistemically irrelevant to each other, and therefore epistemically independent.

CHOICE FUNCTIONS

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Collect in $R(F)$ the inadmissible gambles from F, and in $C(F) := F \setminus R(F)$ the admissible gambles from F. Then C is a choice function, and R a rejection function.

Choice functions

Example 1 If I have a pmf p, then $R(F) = \{f \in F\colon$ there is $g \in F$ such that $E_p(f) < E_p(g)\}$ and $C(F) = F \setminus R(F)$. This is expected utility maximisation.

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Example 2 If I have a coherent set of desirable gambles D, or a partial preference relation $\prec$, then I can use it to make decisions:

$$R(F) = \{f \in F\colon \text{there is } g \in F \text{ such that } f \prec g\}, \qquad \text{(Sen–Walley maximality)}$$

and then $f \prec g \Leftrightarrow f \in R(\{f, g\})$.

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Unlike the first examples, E-admissibility is an instance of non-binary choice: it may happen that

$$f \in C(\{f, g\}), \quad f \in C(\{f, h\}), \quad \text{even though} \quad f \in R(\{f, g, h\}).$$

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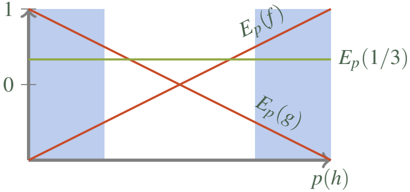

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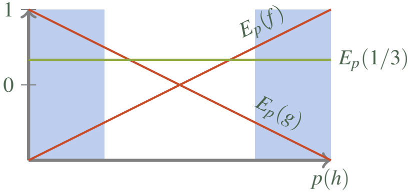

If we use E-admissibility $C(F) = \bigcup_{p \in \mathscr{M}} \{f \in F : E_p(f) \geq E_p(g) \text{ for all } g \in F\}$, then

$$1/3 \in R(\{f, g, 1/3\}) \quad \text{but} \quad 1/3 \in C_v(\{f, g, 1/3\}).$$

Sets of desirable gamble sets

$\mathscr{L}$: linear space of gambles

$D \subseteq \mathscr{L}$: set of desirable gambles

$\mathscr{Q}$: set of gamble sets

$K \subseteq \mathscr{Q}$: set of desirable gamble sets

Let $\mathscr{Q} := \{F \subseteq \mathscr{L} : F \text{ finite}\}$ be the set of all gamble sets.

We assume that $f \in R(F) \Leftrightarrow f - g \in R(F - \{g\})$, where $F - \{g\} := \{f - g : f \in F\}$.

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Given a $F \in \mathscr{Q}$ and $f \in F$, we then have

$$f \in R(F) \Leftrightarrow 0 \in R(F - \{f\})$$

If this happens, we call $F - \{f\}$ a desirable gamble set.

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$$f \in R(F) \Leftrightarrow 0 \in R(F - \{f\}) \Leftrightarrow F - \{f\} \in K.$$

If this happens, we call $F - \{f\}$ a desirable gamble set.

We collect all desirable gamble sets in $K := \{F \in \mathscr{Q} : 0 \in R(F \cup \{0\})\}$. A gamble set F belongs to K if it contains a desirable gamble, but I may not be able to identify which one.

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Conversely, given a set of desirable gamble sets $K \subseteq \mathscr{Q}$, we can define a rejection function R by $f \in R(F) \Leftrightarrow F - \{f\} \in K$ for all $F \in \mathscr{Q}$ and $f \in F$. Both constructions commute.

K and R are equivalent representations of the same information.

Sets of desirable gamble sets: coherence

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We call a set of desirable gamble sets K coherent if

K_0. $\emptyset \notin K$;

K_1. $F \in K \Rightarrow F \setminus \{0\} \in K$;

K_2. $\{f\} \in K$ for all f in $\mathscr{L}_{>0}$;

K_3. if $F, G \in K$ and $(\lambda^{f,g}, \mu^{f,g}) > 0$ for every pair (f,g) in $F \times G$, then $\{\lambda^{f,g} f + \mu^{f,g} g : f \in F, g \in G\} \in K$;

K_4. if $F \in K$ and $F \subseteq G$, then $G \in K$,

for all gamble sets F and G.

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$K_D := \{F \in \mathscr{Q} : F \cap D \neq \emptyset\}$

K is coherent iff $K = \bigcap_{D \in \mathscr{D}} K_D$ for some non-empty $\mathscr{D}$.

If I have a coherent set of desirable gambles D, then I can use it to make decisions, using Sen–Walley maximality:

$$R_D(F) := \{f \in F : \text{there is } g \in F \text{ such that } g - f \in D\} \quad \text{for all } F \in \mathscr{Q},$$

so

$$K_D := \{F \in \mathscr{Q} : 0 \in R_D(F \cup \{0\})\} = \{F \in \mathscr{Q} : \text{there is } f \in F \text{ such that } f \in D\}$$

We use these K_Ds to represent a coherent K.

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We use these K_Ds to represent a coherent K.

Theorem (Representation)

For any sets of desirable gamble sets $K \subseteq \mathscr{Q}$, the following two expressions are equivalent:

(i) K is coherent;

(ii) there is a non-empty set $\mathscr{D}$ of coherent sets of desirable gambles such that $K = \bigcap\{K_D \colon D \in \mathscr{D}\}$. If this is the case, then K is said to be represented by $\mathscr{D}$, and $\mathscr{D}$ is a representation of K.

Moreover, K's (unique) largest representing set is $\mathscr{D}(K) := \{D \text{ coherent} \colon K \subseteq K_D\}$.

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Coherence is a mixture of E-admissibility and Sen–Walley maximality.

Sets of desirable gamble sets: natural extension

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Consider an assessment, or partially specified set of desirable gamble sets $\mathscr{A} \subseteq \mathscr{Q}$. If $\mathscr{A}$ has a coherent superset K, then $\mathscr{A}$ is called consistent.

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Theorem

$\mathscr{A}$ is consistent if and only if $\mathscr{A} \subseteq K_D$ for some coherent D. Moreover, if $\mathscr{A}$ is consistent, then its smallest coherent superset $\mathscr{E}(\mathscr{A})$ – called natural extension – is equal to

$$\begin{aligned}\mathscr{E}(\mathscr{A}) &= \bigcap \{K_D : \mathscr{A} \subseteq K_D, D \text{ a coherent set of desirable gambles}\} \\ &= \bigcap \{K_D : D \in \mathscr{D}(\mathscr{A})\}.\end{aligned}$$

Sets of desirable gamble sets: information

Find more information in the work of:

Teddy Seidenfeld

Gert de Cooman

Arthur Van Camp

Enrique Miranda

Jasper De Bock

Catrin Campbell-Moore

Kevin Blackwell

Jason Konek

Arne Decadt

Alexander Erreygers

INDEPENDENCE

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We say that a coherent K satisfies epistemic independence if

$$\mathrm{marg}_Y(K \mid E_X) = \mathrm{marg}_Y(K) \quad \text{and} \quad \mathrm{marg}_X(K \mid E_Y) = \mathrm{marg}_X(K)$$

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Theorem (Epistemically independent natural extension)

If K_X is a coherent set of desirable gamble sets about X, and K_Y is a coherent set of desirable gamble sets about Y, then the smallest epistemic product of K_X and K_Y exists, denoted by $K_X \otimes K_Y$.

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The idea of representational independence is:

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Theorem (Representationally independent natural extension)

If K_X is a coherent set of desirable gamble sets about X, and K_Y the same but about Y, then the smallest representational product of K_X and K_Y exists, and is given by

$$K_X \boxdot K_Y := \bigcap_{D_X \in \mathscr{D}(K_X), D_Y \in \mathscr{D}(K_Y)} K_{D_X \otimes D_Y}.$$

S-independence: definition

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The idea of S-irrelevance (Seidenfeld-irrelevance) is:

You should not spend resources in order to use the observed value of Y to choose between gambles that only depend on X.

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Consider a partition $\mathscr{P}$ of $\mathscr{X}$, and for each $E \in \mathscr{P}$, a gamble f_E on $\mathscr{Y}$. The action of first observing X and then selecting a gamble on $\mathscr{Y}$ corresponds to the composite gamble $\sum_{E \in \mathscr{P}} I_E(X) f_E(Y)$, and paying the price $\varepsilon > 0$ for it is the gamble

$$f := \sum_{E \in \mathscr{P}} I_E(X) f_E(Y) - \varepsilon.$$

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The requirement of S-irrelevance is that $f \in R(\{f, f_E \colon E \in \mathscr{P}\})$, or, after subtracting f:

$$\left\{ \sum_{G \in \mathscr{P} \setminus \{E\}} I_G(f_E - f_G) + \varepsilon \colon E \in \mathscr{P} \right\} \in K,$$

for every partition $\mathscr{P}$ of $\mathscr{X}$, and all choices of the gambles f_E on $\mathscr{Y}$ and real $\varepsilon > 0$. S-independence is requiring S-irrelevance in both directions.

S-independence

$\mathscr{L}$: linear space of gambles

$D \subseteq \mathscr{L}$: set of desirable gambles

$\mathscr{Q}$: set of gamble sets

$K \subseteq \mathscr{Q}$: set of desirable gamble sets

$K_D := \{F \in \mathscr{Q} : F \cap D \neq \emptyset\}$

K is coherent iff $K = \bigcap_{D \in \mathscr{D}} K_D$ for some non-empty $\mathscr{D}$.

A coherent K about (X, Y) is called an S-product of two coherent marginals K_X about X and K_Y about Y if it satisfies S-independence and has K_X and K_Y as marginals.

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Theorem (S-independent natural extension)

Consider Archimedean and mixing K_X and K_Y that have positive probabilities for every outcome. Then the smallest S-independent product of K_X and K_Y exists, and is given by

$$K_X \circledast K_Y := \bigcap_{P_X \in \underline{\mathscr{P}}(K_X), P_Y \in \underline{\mathscr{P}}(K_Y)} K_{D_{P_X P_Y}}$$

Mixingness is the requirement
K_5. If $G \in K$ and $F \subseteq G \subseteq \operatorname{posi} F$, then also $F \in K$.
A mixing K is represented by a set of mixing Ds, that is, D for which $\operatorname{posi}(D^c) = D^c$.

Archimedeanity has as a consequence that K is represented by a set $\underline{\mathscr{P}}$ of lower previsions.

Therefore, a mixing and Archimedean K is represented by a set of linear previsions.

Representational independence vs S-independence

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Proposition

If K is Archimedean, has positive probabilities for every outcome and satisfies S-independence, then it satisfies representational independence.

The idea is that such K is mixing!

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Proposition

If K is mixing and satisfies representational independence, then it satisfies S-independence.

DISCUSSION

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2. What properties do we want from an independence definition?

2a. For use in credal networks, we need the corresponding independent product to be associative.

2b. Ideally, we want the representation to be of the same type. For instance:

$$K_X \boxdot K_Y := \bigcap_{D_X \in \mathscr{D}(K_X), D_Y \in \mathscr{D}(K_Y)} K_{D_X \otimes D_Y},$$

so the $K_X \boxdot K_Y$ is represented by $\{D_X \otimes D_Y : D_X \in \mathscr{D}(K_X), D_Y \in \mathscr{D}(K_Y)\}$. This is not equal to $\mathscr{D}(K_X \boxdot K_Y)$, even though the 'inputs' are $\mathscr{D}(K_X)$ and $\mathscr{D}(K_Y)$.

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2c. We want it to generalise the standard definition of independence for pmfs. Fortunately, for any pmfs p on $\mathscr{X}$ and q on $\mathscr{Y}$ with positive probabilities

$$K_p \otimes K_q = K_p \boxcircle K_q = K_p \circledast K_q = K_{pq},$$

so the three concepts of independence coincide for precise probabilities.

QUESTIONS?