Learning from Preferences and Choice functions

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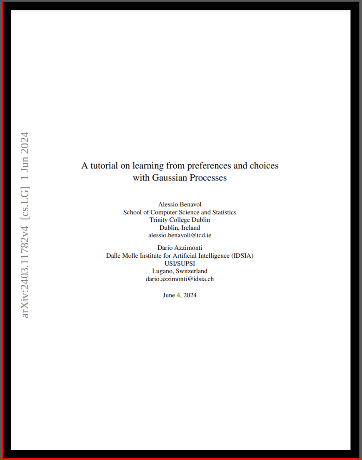
Background



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A tutorial on learning from preferences and choices
with Gaussian Processes

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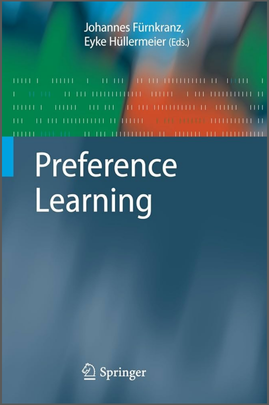


Outline

Learning from Human Feedback

Object Preference

Consistent Preferences
Less rationality
- discernibility (Luce)
- noise (Thurstone)
More rationality
- Monotonicity

Label Preference

- Plackett-Luce Model
- Thurstone's Model
- Bradley-Terry Model

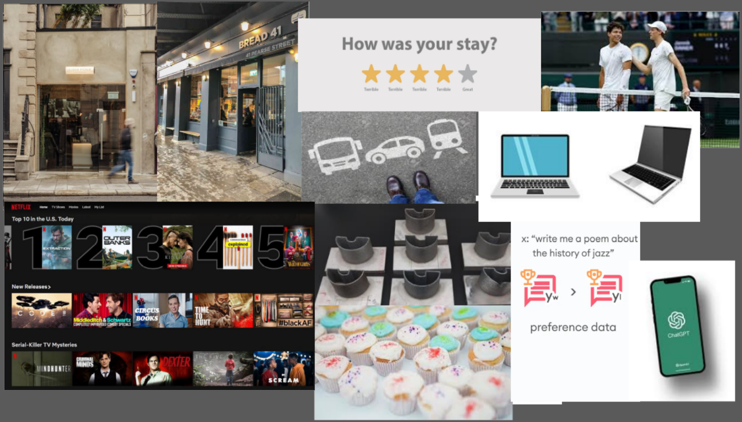
Choice functions

- Pareto rationality
- quasi-rationality

Utility function

Two-argument function

Preferences



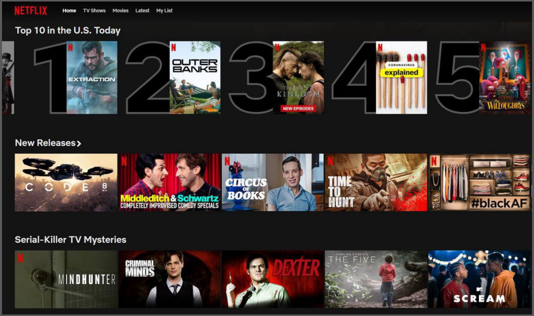
PL aims to learn preference models from observed, revealed or automatically extracted preference information.

Object Preference

Laptop Model	Laptop 1	Laptop 2
Processor	Intel I7	Intel I7
Generation	10th	11th
Operating System	Windows	MacOS

- a set of objects $\{k |\ k = 1, \ldots, m\}$
- a vector of characteristics for each objects $\mathbf{x}_k \in \mathbb{R}^q$
  - $\mathcal{X} = \{\mathbf{x}_k :\ \ k = 1, \ldots, m\}$
- an associated set of pairwise preferences of the form $\mathbf{x}_i \succ \mathbf{x}_j$

Label Preference



- a set of instances $\mathcal{X} = \{\mathbf{x}_k : \; k = 1, \ldots, m\}$
- a set of labels $\mathcal{Y} = \{y_1, \ldots, y_\ell\}$
- for each instance $\mathbf{x}_k$ an associated set of pairwise preferences of the form $y_i \succ_{\mathbf{x}_k} y_j$

What is a (strict) preference?

Asymmetry

If  $\succ$  then  $\nsucc$ 

Negative Transitivity

If  $\succ$  then for any 

either  $\succ$  or  $\succ$  or both

A strict preference relation is said to be *consistent* – *rational* – when it satisfies the above two properties.

Utility representation

Definition

For a preference relation $\succ$ on $\mathcal{X}$, the function $u : \mathcal{X} \to \mathbb{R}$ represents $\succ$ if

$$\mathbf{x}_i \succ \mathbf{x}_j \quad \text{if} \quad u(\mathbf{x}_i) > u(\mathbf{x}_j). \tag{1}$$

We say that u is a utility function for $\succ$.

Example

utility=taste

utility=performance

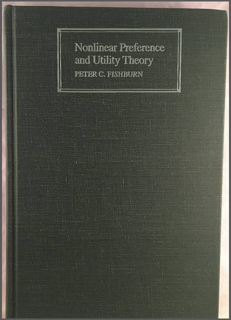
Theorem (Debreu 1954)

The relation $\succ$ admits a utility function representation iff it is **consistent** *(asymmetric and negatively transitive)*

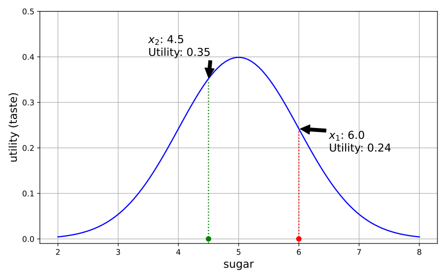
Two-argument function representation

Definition

$$\mathbf{x}_i \succ \mathbf{x}_j \quad \textit{if} \quad q(\mathbf{x}_i, \mathbf{x}_j) > 0 \tag{2}$$



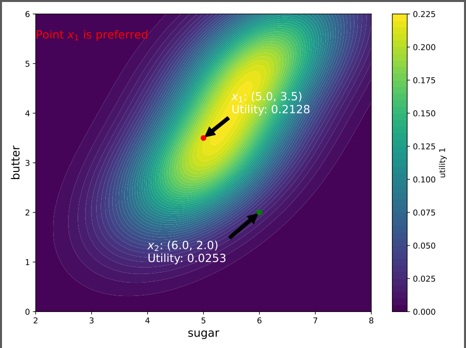
Example



4.5  $\succ$ 6 

$\Big\{$ 4.5  $\succ$ 6 , 4  $\succ$ 2.5 , ... $\Big\} \Rightarrow u(sugar)$

Example



(5,3.5)  $\succ$ (6,2) 

$\Bigg\{$ (5,3.5)  $\succ$ (6,2)  $, \ldots \Bigg\} \Rightarrow u(sugar, butter)$

Object preference learning

1. Given $\mathcal{D}_m = \{x_1 \succ x_3, x_4 \succ x_2, x_5 \succ x_1, \ldots\}$
2. Predict $x_2 \overset{?}{\succ} x_3$

Object PL: Parametric

From Debreu's theorem, we can derive that

$$\mathcal{D}_m = \{\mathbf{x}_l^{(s)} \succ \mathbf{x}_r^{(s)} : \;\; s = 1, \ldots, m\} = \{u(\boldsymbol{x}_l^{(s)}) - u(\boldsymbol{x}_r^{(k)}) > 0 : \;\; s = 1, \ldots, m\}$$

Functional form for $u(\mathbf{x})$:

Linear [Har-Peled et al., 2002]: $u(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$

Log-Linear [Dekel et al., 2003]: $\ln u(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$

Non-linear (NNs): $u(\mathbf{x}) = \mathbf{w}^\top \mathbf{b}_\theta(\mathbf{x})$

The unknowns are determined by either solving a **constrained optimization** problem or by minimizing a loss function. No uncertainty representation.

Object PL: Non-parametric (Gaussian Process)

$$\mathcal{D}_m = \{\mathbf{x}_l^{(s)} \succ \mathbf{x}_r^{(s)} : \;\; s = 1, \ldots, m\} = \{u(\boldsymbol{x}_l^{(s)}) - u(\boldsymbol{x}_r^{(k)}) > 0 : \;\; s = 1, \ldots, m\}$$

By defining the objects as a matrix $X = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_r]^\top$ and the vector $\mathbf{u}(X) = [u(\boldsymbol{x}_1), u(\boldsymbol{x}_2), \ldots, u(\boldsymbol{x}_r)]^\top$, we can write these m constraints as:

$$W\mathbf{u}(X) > 0$$

Likelihood: $p(\mathcal{D}_m|\mathbf{u}(X)) = I_{\{W\mathbf{u}(X)>0\}}(\mathbf{u}(X))$

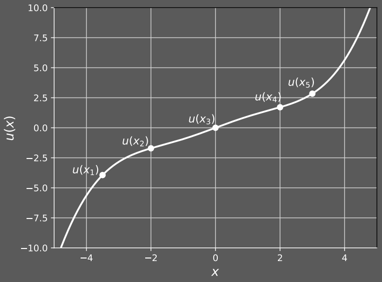
Prior: $p(\mathbf{u}(X)) = N(\mathbf{u}(X); \mathbf{0}, K_{\boldsymbol{\theta}}(X, X))$

Posterior: $p(\mathbf{u}(X)|\mathcal{D}_m) = TN_{\{W\mathbf{u}(X)>0\}}(\mathbf{u}(X); \mathbf{0}, K_{\boldsymbol{\theta}}(X, X))$

Posterior pred.: $p(\mathbf{u}(X^*)|\mathcal{D}_m) = \int p(\mathbf{u}(X^*)|\mathbf{u}(X))\, p(\mathbf{u}(X)|\mathcal{D}_m) d\mathbf{u}(X)$

What is a GP?

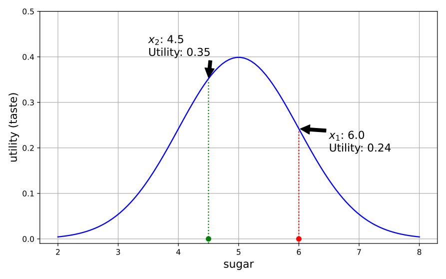
$$u \sim GP(m(x), k(x, x'))$$

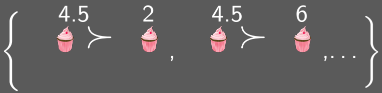


$$\begin{bmatrix} u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \\ u(x_5) \end{bmatrix} \sim N\left(\begin{bmatrix} m(x_1) \\ m(x_2) \\ m(x_3) \\ m(x_4) \\ m(x_5) \end{bmatrix}, \begin{bmatrix} k(x_1, x_1) & k(x_1, x_2) & \dots & k(x_1, x_5) \\ k(x_2, x_1) & k(x_2, x_2) & \dots & k(x_2, x_5) \\ \vdots & \vdots & \vdots & \vdots \\ k(x_5, x_1) & k(x_5, x_2) & \dots & k(x_5, x_5) \end{bmatrix} \right)$$

Typical Kernel: $k_{\boldsymbol{\theta}}(\mathbf{x}, \mathbf{x}') = \exp\left(-\sum_{i=1}^{c} \frac{(x_i - x_i')^2}{2\ell_i^2} \right)$

Example

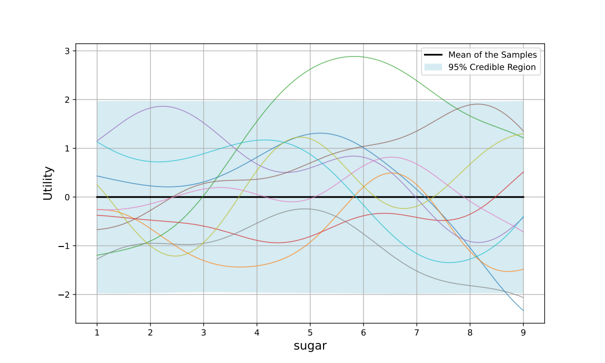




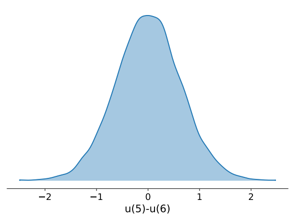
$\Rightarrow u(sugar)$ and predict $u(5) \overset{?}{>} u(6)$

Prior

$$\text{Prior:} \quad p(\mathbf{u}(x)) = GP(m(x), k(x, x'))$$

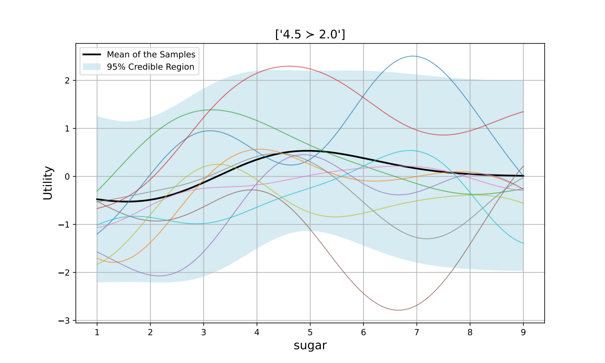


$$u(5) \overset{?}{>} u(6)$$

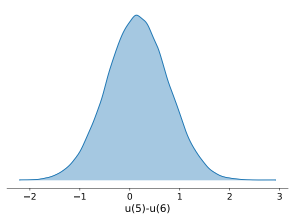


One preference

Posterior: $p(\mathbf{u}(X)|\mathcal{D}_m) = TN_{\{W\mathbf{u}(X)>0\}}(\mathbf{u}(X); \mathbf{0}, K_\theta(X, X))$

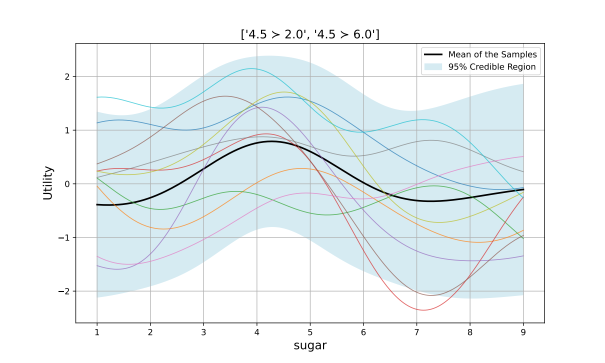


$u(5) \overset{?}{>} u(6)$

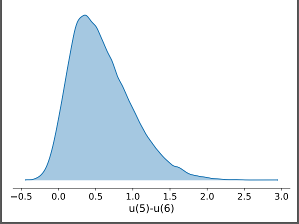


Two preferences

Posterior: $p(\mathbf{u}(X)|\mathcal{D}_m) = TN_{\{W\mathbf{u}(X)>0\}}(\mathbf{u}(X); \mathbf{0}, K_\theta(X,X))$

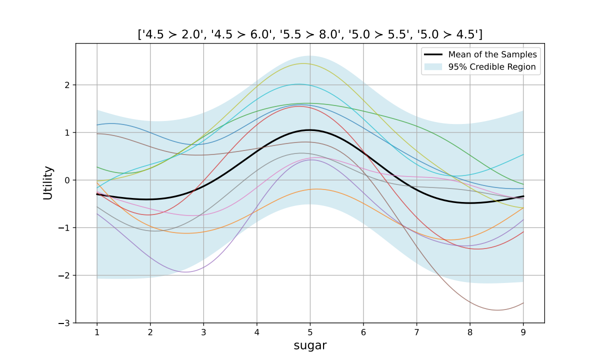


$u(5) \overset{?}{>} u(6)$

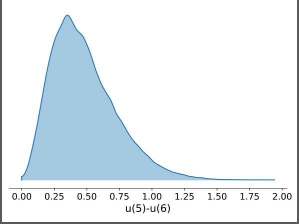


Five preferences

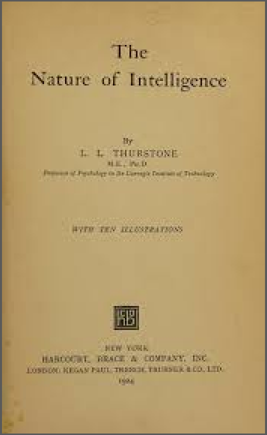
Posterior: $p(\mathbf{u}(X)|\mathcal{D}_m) = TN_{\{W\mathbf{u}(X)>0\}}(\mathbf{u}(X); \mathbf{0}, K_\theta(X,X))$

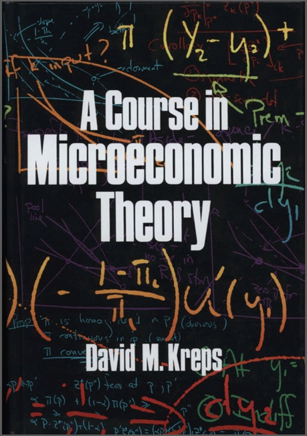


$u(5) \overset{?}{>} u(6)$



The Nature of Intelligence





Translating these ideas into statistical learning!

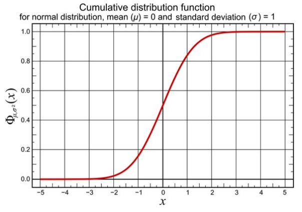
User's preferences may be inconsistent

1. Limit of discernability:

2. Noise in the observed utility

Robustify the model: Limit of discernability

$$\begin{cases} \mathbf{x}_i \succ \mathbf{x}_j & \text{with probability } \Phi\left(\frac{u(\mathbf{x}_i)-u(\mathbf{x}_j)}{\sigma}\right) \\ \mathbf{x}_j \succ \mathbf{x}_i & \text{with probability } 1-\Phi\left(\frac{u(\mathbf{x}_i)-u(\mathbf{x}_j)}{\sigma}\right) \end{cases}$$



Likelihood: $p(\mathcal{D}_m|\mathbf{u}(X)) = \prod_{s=1}^{m} \Phi\left(\frac{u(\mathbf{x}_l{}^{(s)})-u(\mathbf{x}_r{}^{(s)})}{\sigma}\right) = \Phi_m\left(\frac{W}{\sigma}\mathbf{u}(X)\right)$

Prior: $p(\mathbf{u}(X)) = N(\mathbf{u}(X); \mathbf{0}, K_\theta(X,X))$

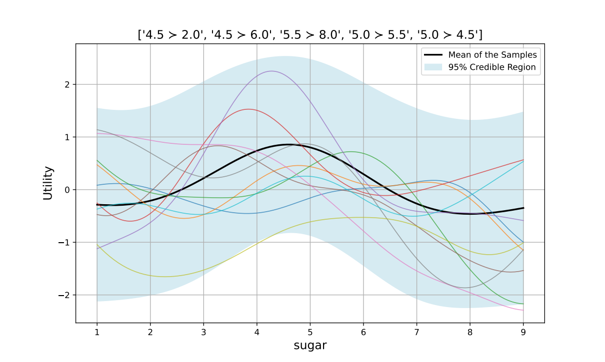
Posterior: $p(\mathbf{u}(X)|\mathcal{D}_m) = \text{SkewNormal}(\mathbf{u}(X); \mathbf{0}, K_\theta(X,X), \Delta, \Gamma, \gamma)$

Posterior pred.: $p(\mathbf{u}(X^*)|\mathcal{D}_m) = \int p(\mathbf{u}(X^*)|\mathbf{u}(X))\, p(\mathbf{u}(X)|\mathcal{D}_m) d\mathbf{u}(X)$

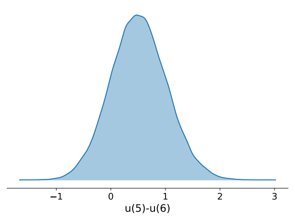
This model was derived in [Benavoli et al., 2021a, Benavoli et al., 2021b]. Originally proposed by [Chu and Ghahramani, 2005] using Laplace's approximation.

Five preferences

Posterior: $p(\mathbf{u}(X)|\mathcal{D}_m) = \text{SkewNormal}(\mathbf{u}(X); \mathbf{0}, K_{\theta}(X,X), \Delta, \Gamma, \gamma)$



$u(5) \overset{?}{>} u(6)$



User's preferences may be inconsistent due to time

- **Monday**:

 $\succ$ 

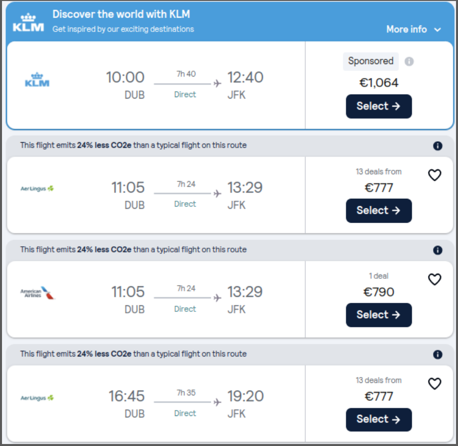
- **Tuesday**:

 $\succ$ 

Make the model to be aware of time:

$$\text{Prior:} \quad p(\mathbf{u}(X,t)) = N(\mathbf{u}(X,t); \mathbf{0}, K_{\boldsymbol{\theta}}((X,t),(X,t)))$$

More rationality



Objects are flights:

$$\mathbf{x}_i = \begin{bmatrix} \text{cost} \\ \text{duration} \\ \text{company} \\ \text{dep. time} \end{bmatrix}$$

Additional rationality constraints:

$$\begin{bmatrix} \text{cost} + \delta \\ \text{duration} \\ \text{company} \\ \text{dep. time} \end{bmatrix} \prec \begin{bmatrix} \text{cost} \\ \text{duration} \\ \text{company} \\ \text{dep. time} \end{bmatrix}$$

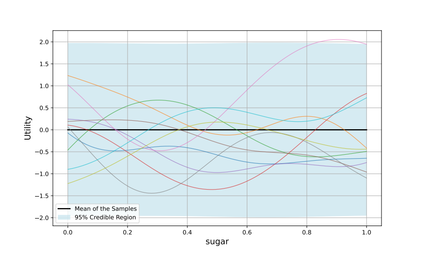
or, equivalently,

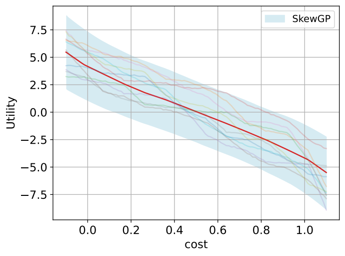
$$u(\text{cost} + \delta, \dots) < u(\text{cost}, \dots)$$

Similarly for *duration*.

Monotonic GP

$$u(cost) \sim GP(m(cost), k(cost, cost')) \quad \text{subject to} \quad \frac{d}{d\text{cost}} u < 0$$





When gambles are the objects

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \succ \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then there is a utility that represents the preference

$$u(\mathbf{x}_1) > u(\mathbf{x}_2)$$

Add monotonicity

$$\frac{\partial}{\partial x_{11}} u(\mathbf{x}_1) > 0, \quad \frac{\partial}{\partial x_{12}} u(\mathbf{x}_1) > 0$$

Given the preferences

$$\{\mathbf{x}_1 \succ \mathbf{x}_2, \quad \mathbf{x}_1 \succ \mathbf{x}_4 \ldots\} \quad \text{learn} \quad u$$

When gambles are the objects

$\succ$

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \succ \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then there is a utility that represents the preference

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Add monotonicity

$$\frac{\partial}{\partial x_{11}} u(\mathbf{x}_1) > 0, \quad \frac{\partial}{\partial x_{12}} u(\mathbf{x}_1) > 0$$

Given the preferences

$$\{\mathbf{x}_1 \succ \mathbf{x}_2, \quad \mathbf{x}_1 \succ \mathbf{x}_4 \ldots\} \quad \text{learn} \quad u$$

Add linearity

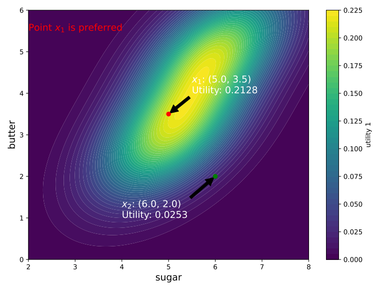
$$u(\mathbf{x}_1) = w_1 x_{11} + w_2 x_{12} \quad \text{equiv.} \quad u(\mathbf{x}) \sim GP(0, \mathbf{x}\mathbf{x}^\top)$$

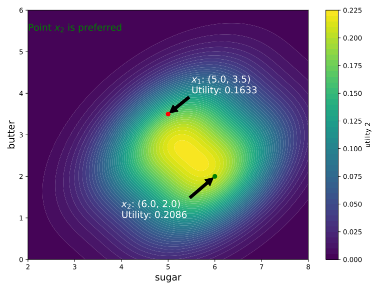
Multiple utilities and incompleteness

(**cpu**,drive)  ≻ (**cpu**,drive) 

(cpu,**drive**)  ≻ (cpu,**drive**) 

(**cpu**,drive)  ≻ (**cpu**,drive) 





Choice functions

$A = \{$  ,  ,  ,  $\}$, $C(A) = \{$  ,  $\}$

- rationality criteria for choices (Chernoff, Expansion, Aizerman, Sen-α..)

  $A = \{$  ,  ,  $\}$, $C(A) = \{$  ,  $\}$
- utility representation
  - A cupcake is selected if it is the best with respect to (at least) one utility (in IP e-admissibility)

Choice functions

$A = \{$  ,  ,  ,  $\}$, $\quad C(A) = \{$  ,  $\}$

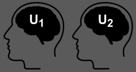
- rationality criteria for choices (Chernoff, Expansion, Aizerman, Sen-α..)

  $A = \{$  ,  ,  $\}$, $\quad C(A) = \{$  ,  $\}$

- utility representation
  - A cupcake is selected if it is the best with respect to (at least) one utility (in IP e-admissibility)

$$p(C(A), A|u_1(X), \ldots, u_L(X)) = \prod_{\{\mathbf{o},\mathbf{v}\} \in C_\sharp(A)} \left(1 - \prod_{i=1}^{L} I_{u_i(\mathbf{o})>u_i(\mathbf{v})}(\mathbf{u}(X)) - \prod_{i=1}^{L} I_{u_i(\mathbf{v})>u_i(\mathbf{o})}(\mathbf{u}(X))\right)$$

$$\prod_{\mathbf{v} \in R(A)} \left(\prod_{i=1}^{L} \left(1 - \prod_{\mathbf{o} \in C(A)} I_{u_i(\mathbf{v})>u_i(\mathbf{o})}(\mathbf{u}(X))\right)\right)$$



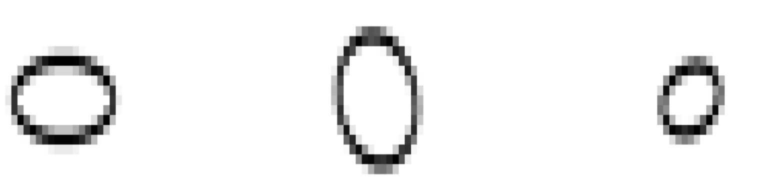
$u_1 \sim GP(0, k)$

$u_2 \sim GP(0, k)$

Example: choosing ellipses

Two criteria:

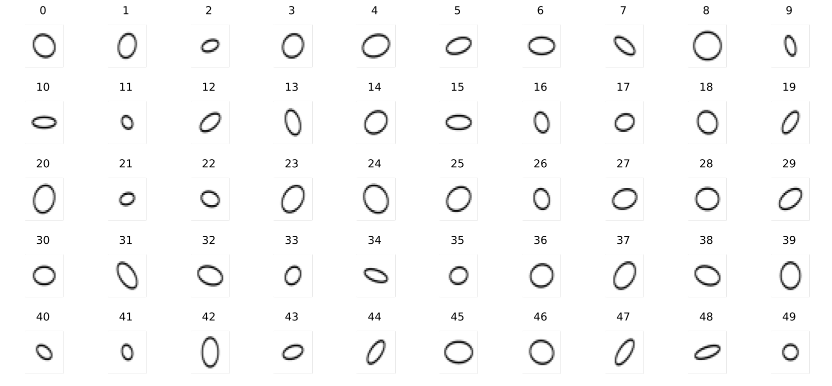
- "being closer to a circle"
- "being aligned to the axes"



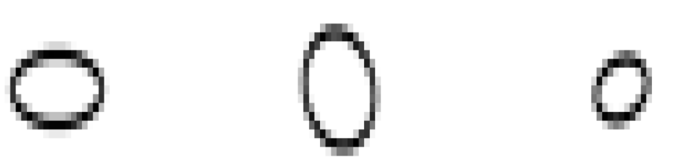
The student selected the left and right-ellipse.

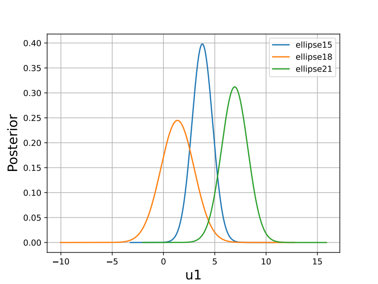
Example: choosing ellipses

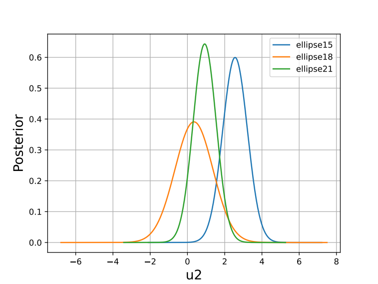
Dataset: 50 images 25×25 pixels and 160 choices



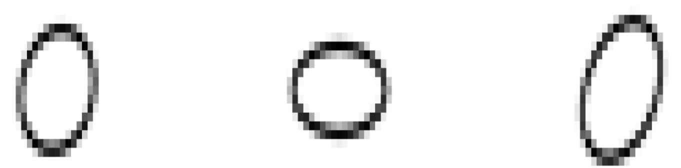
Example: choosing ellipses

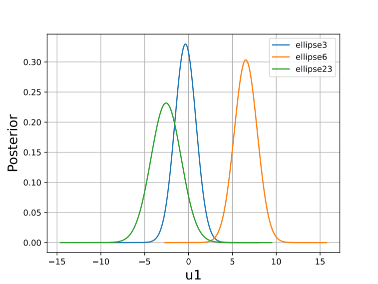


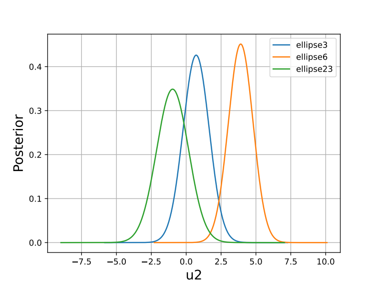




Example: choosing ellipses

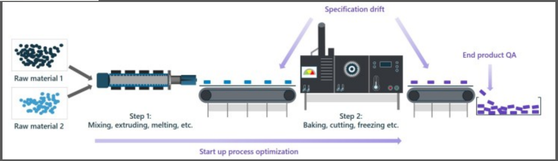




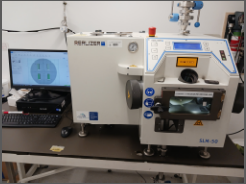
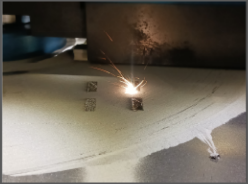
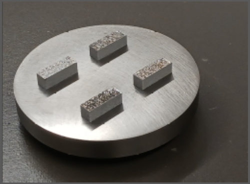


I could make ellipses that the model predicts you would like the most.

Smart manufacturing



Smart manufacturing

Block 0 $\mathbf{x}_0 = [\mathit{Power},\ \mathit{Speed},\ \mathit{HatchSpacing}] = [300, 100, 40]$

Block 1 $\mathbf{x}_1 = [90, 700, 140]$

Block 2 $\mathbf{x}_2 = [70, 250, 50]$

Block 3 $\mathbf{x}_3 = [40, 550, 110]$

Thank you

Preferences are everywhere!

desirability

$$u(x_i) > 0$$

preference

$$u(x_i) > u(x_j)$$

ranking

$$\tau_{2star} > u(x_i) > \tau_{1star}$$

interval

$$7 > u(x_i) > 6$$

precise measurements

$$6.3 + \epsilon > u(x_i) > 6.3$$

Thank you

Preferences are everywhere!

desirability

$u(x_i) > 0$

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interval

$7 > u(x_i) > 6$

precise measurements

$6.3 + \epsilon > u(x_i) > 6.3$

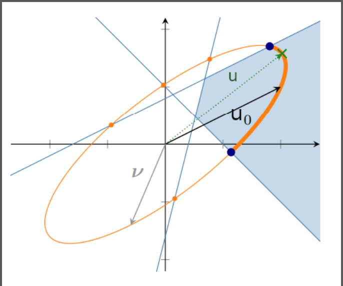
Linear elliptical slice sampling [Gessner et al., 2020]

1. Given previous sample $u_0(X)$
2. Sample $\nu(X) \sim N(\mathbf{u}(X); \mathbf{0}, K_{\boldsymbol{\theta}}(X, X))$
3. Define the ellipse

$$u_0(X)\cos(\phi) + \nu(X) sin(\phi), \quad \phi \in [0, 2\pi)$$

4. Intercept ellipse with $\{W\mathbf{u}(X) > 0\}$
5. Sample ϕ at random inside the intersection

$$u_1(X) = u_0(X)\cos(\phi) + \nu(X) sin(\phi)$$



Learning Hyperparameters

Marginal likelihood:

$$\arg\max_{\boldsymbol{\theta}} p(\mathcal{D}_m|\boldsymbol{\theta}) = \int I_{\{W\mathbf{u}(X)>0\}}(\mathbf{u}(X))N(\mathbf{u}(X);\mathbf{0},K_{\boldsymbol{\theta}}(X,X))d\mathbf{u}(X)$$

Approximation:

1. Approximation of numerical integration for Truncated Gaussian;
2. Laplace's approximation;
3. Variational approximation.

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