Information algebra of sets of desirable gamble sets

compatibility and beyond

1 Information algebra

ⓘ An **information algebra** is a mathematical framework for modeling how information is represented, combined, and focused within a system.

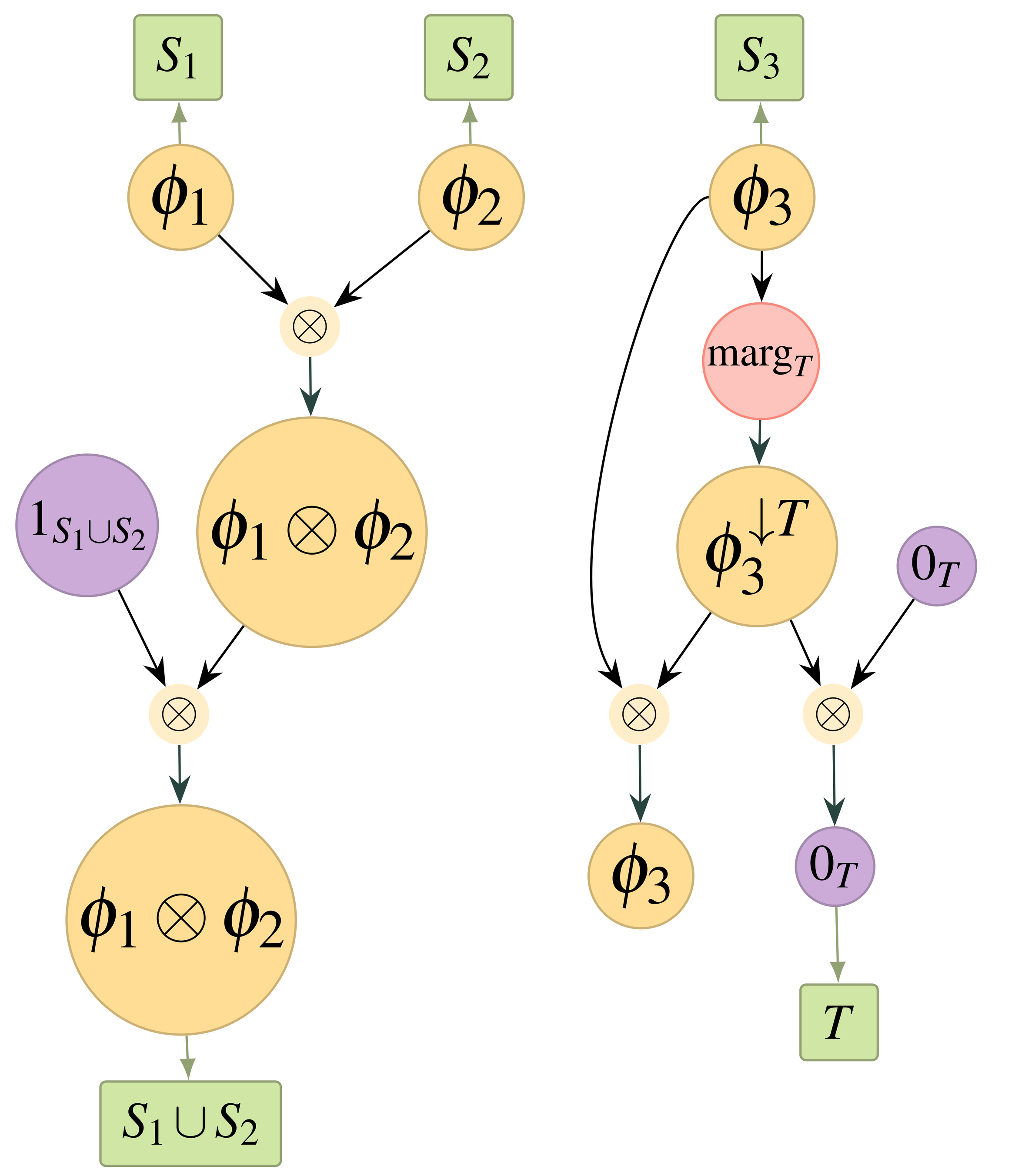
Vocabulary	
$\Omega_S \subseteq \Omega_N = \Omega$	possibility space for random variables taking values in S
$S \subseteq N$	subset of indices of possibility space
$\mathfrak{D} = \{S : S \subseteq N\}$	lattice of indices
$\phi, \psi, \ldots$	pieces of information
$\Phi_S \subseteq \Phi$	all information about Ω_S
$\Phi := \bigcup_{S \subseteq N} \Phi_S$	the set of all information

Information algebras admit three operations:

$d : \Phi \longrightarrow \mathfrak{D} : d(\phi) = S$ if $\phi \in \Phi_S$ **labeling**

$\otimes : \Phi \times \Phi \longrightarrow \Phi : (\phi, \psi) \mapsto \phi \otimes \psi$ **combination**

$\downarrow : \Phi \times \mathfrak{D} \longrightarrow \Phi : (\phi, S) \mapsto \phi^{\downarrow S}$ **marginalization**



ⓘ For all $S \subseteq N$, the unit element 1_S represents **vacuous information**; in turn, the null element 0_S represents **contradiction**.

AXIOMS

1. $(\Phi, \otimes)$ is associative and commutative. For all ϕ with $d(\phi) = S$, there are elements 1_S and 0_S such that $1_S \otimes \phi = \phi$ and $0_S \otimes \phi = 0_S$. **[COMMUTATIVE MONOID WITH NULL]**
2. $d(\phi \otimes \psi) = d(\phi) \cup d(\psi)$. **[LABELING]**
3. $d(\phi^{\downarrow S}) = S$. **[MARGINALIZATION]**
4. $d((\phi^{\downarrow T})^{\downarrow S}) = S$. **[TRANSITIVITY]**
5. $(\phi \otimes \psi)^{\downarrow S} = \phi \otimes \psi^{\downarrow S \cap T}$ if $d(\phi) = S$ and $d(\psi) = T$. **[COMBINATION]**
6. $1_S \otimes 1_T = 1_{S \cup T}$ and $0_S \otimes 0_T = 0_{S \cup T}$. **[NEUTRALITY]**
7. $\phi \otimes \phi^{\downarrow S} = \phi$ for $S \subseteq d(\phi)$. **[IDEMPOTENCY]**

2 Algebra of sets of desirable gambles

Consider coherent sets of desirable gambles $\overline{\mathscr{D}}(\Omega)$, where $\Omega = \Omega_N$ is a Cartesian domain of N random variables.

Algebra of sets of desirable gambles	
$D, D_1, \cdots \in \overline{\mathscr{D}}(\Omega_S)$	pieces of information
$\overline{\mathscr{D}}(\Omega_S)$	all information about Ω_S
$d(D) = S$	labeling
$D_1 \otimes D_2 =$ posi$(D_1 \cup D_2 \cup \mathscr{L}_{>0}(\Omega_{S \cup T}))$	combination (natural extension)
marg$_T D = D \cap \mathscr{L}(\Omega_T)$	marginalization
$\mathscr{L}_{>0}(\Omega_S)$	unit element of Ω_S
$\mathscr{L}(\Omega_S)$	null element of Ω_S

ⓘ **Coherent sets of desirable gambles form an information algebra.**

E. Miranda, M. Zaffalon, *Compatibility, desirability, and the running intersection property*

3 Marginal problem for information algebras

A family of coherent sets of desirable gambles $D_1, \ldots, D_n$ with $d(D_i) = S_i$ is called **compatible** if there is a coherent D such that marg$_{S_i} D = D_i$.

$D_1 \subseteq \overline{\mathscr{D}}(\Omega_{S_1})$ and $D_2 \subseteq \overline{\mathscr{D}}(\Omega_{S_2})$ are **pairwise compatible** if marg$_{S_1 \cap S_2} D_1 =$ marg$_{S_1 \cap S_2} D_2$.

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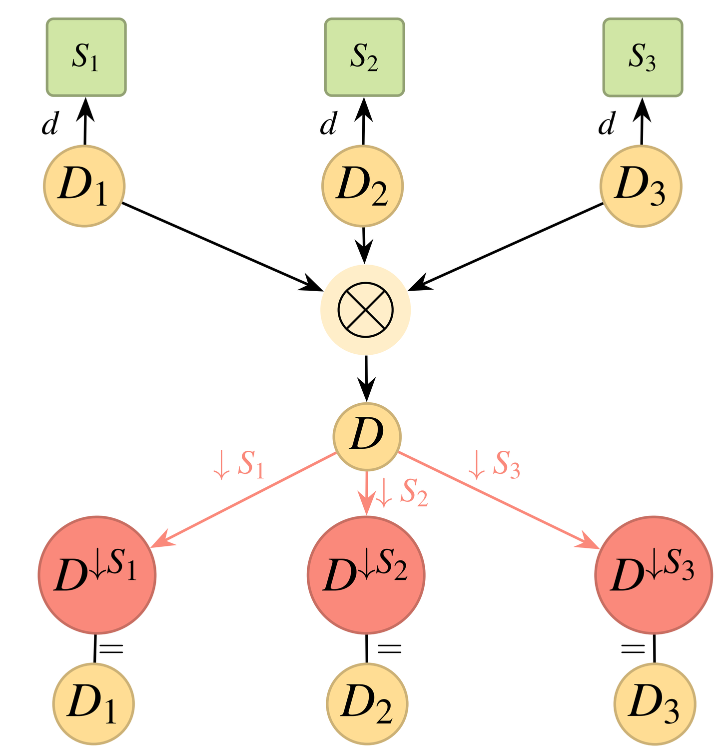
The index sets $S_1, \ldots, S_m$ satisfy the **running intersection property** when

$$(\forall \ell \in \{2, \ldots, m\})(\exists i^\star < \ell) \quad S_\ell \cap S_{i^\star} = S_\ell \cap \bigcup_{i<\ell} S_i.$$

⇓

If $D_1, \ldots, D_n$ are pairwise compatible and $S_1, \ldots, S_n$ satisfy RIP then they are **compatible** with a joint $D = \bigotimes_{i=1}^n D_i$.



With compatibility, we can retrieve the exact information that contributed to the joint.

ⓘ **This result for compatibility follows for all information algebras.**

A. Casanova, J. Kohlas, M. Zaffalon, *Information algebras in the theory of imprecise probabilities*

4 Information algebras for choice function theory (research question as a commutative diagram)

Necessary definitions

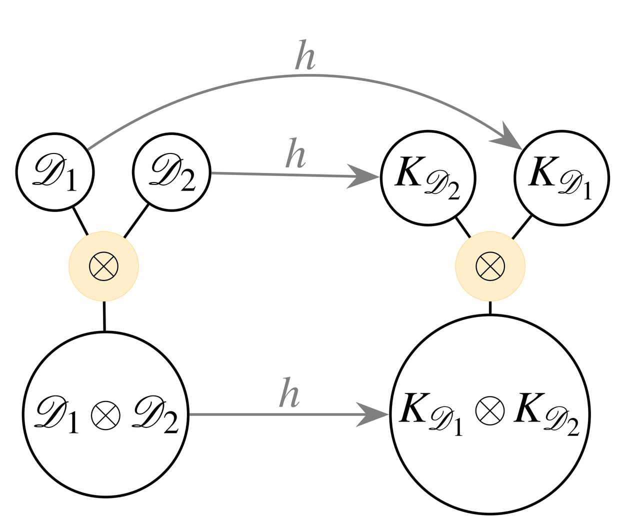
ⓘ Homomorphism

$h(\phi \otimes \psi) = h(\phi) \otimes h(\psi)$, **combination**

$h(\phi^{\downarrow S}) = h(\phi)^{\downarrow S}$, **marginalization**

$h(1_S) = 1_S^\star$ and $h(0_S) = 0_S^\star$. **unity and null**

If h is bijective and one-to-one, it is an **isomorphism**.



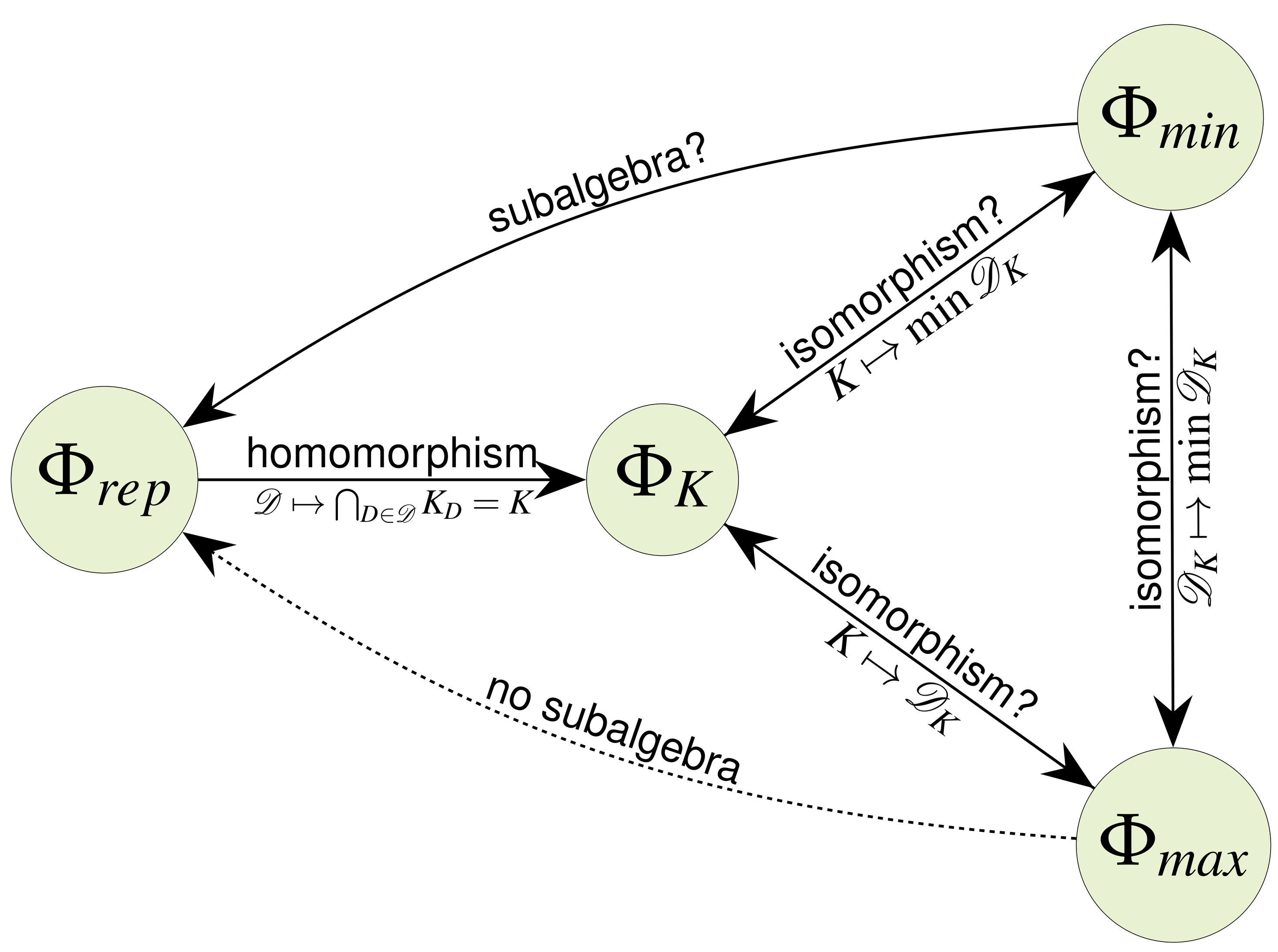
There is a homomorphism $\Phi_{rep} \to \Phi_K$.

ⓘ Subalgebra $\Phi' \subseteq \Phi$

$\phi, \psi \in \Phi'$ then $\phi \otimes \psi \in \Phi'$, **combination**

$\phi \in \Phi'$ then $\phi^{\downarrow T} \in \Phi'$, **marginalization**

$1_S \in \Phi'$ and $0_S \in \Phi'$. **unity and null**



Φ_K collects sets of desirable gamble sets K.

$K, K_1, \ldots \in \overline{\mathscr{K}}(\Omega_S)$	pieces of information
$d(K) = S$	labeling
marg$_T K = K \cap \mathscr{Q}(\Omega_T)$	marginalization
$K_1 \otimes K_2 =$ Rs(Posi$(K_1 \cup K_2 \cup \mathscr{L}_{>0}^{sing}(\Omega_{S \cup T}))$)	combination
Rs$(\mathscr{L}_{>0}^{sing}(\Omega_S)) = K_v(\Omega_S)$	unit element of Ω_S
$\{\mathscr{Q}(\Omega_S)\}$	null element of Ω_S

Φ_{rep} collects representations $\mathscr{D}$ of sets of desirable gamble sets.

$\mathscr{D}, \mathscr{D}_1, \ldots \in \mathscr{P}(\overline{\mathscr{D}}(\Omega_S))$	pieces of information
$d(\mathscr{D}) = d(\bigcap_{D \in \mathscr{D}} K_D) = d(K) = S$	labeling
marg$_T \mathscr{D} = \{D \cap \mathscr{L}(\Omega_T) : D \in \mathscr{D}\}$	marginalization
$\mathscr{D}_1 \otimes \mathscr{D}_2 = \{D_1 \otimes D_2 : D_1 \in \mathscr{D}_1, D_2 \in \mathscr{D}_2\}$	combination
$\{\mathscr{L}_{>0}(\Omega_S)\}$	unit element of Ω_S
$\{\mathscr{L}(\Omega_S)\}$	null element of Ω_S

Φ_{max} collects largest representations $\mathscr{D}_K$.

$\mathscr{D}_K, \mathscr{D}_{K_1}, \ldots \in \mathscr{P}(\overline{\mathscr{D}}(\Omega_S))$	pieces of information
$\overline{\mathscr{D}}(\Omega_S)$	unit element of Ω_S
$\{\mathscr{L}(\Omega_S)\}$	null element of Ω_S

Φ_{min} collects minimal elements of largest representations $\min \mathscr{D}_K$.

$\min \mathscr{D}_K, \ldots \in \mathscr{P}(\overline{\mathscr{D}}(\Omega_S))$	pieces of information
$\{\mathscr{L}_{>0}(\Omega_S)\}$	unit element of Ω_S
$\{\mathscr{L}(\Omega_S)\}$	null element of Ω_S

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