

Context

How can we  
robustify a probability  
measure  $P_0$ ?

### 1. Distortion of probabilities

#### Neighbourhood models

$d$ : distorting function  
comparing probabilities  
 $\delta$ : distorting factor

$$B_d^\delta(P_0) = \{P \mid d(P, P_0) \leq \delta\}$$

#### Increasing Transformations

$g: [0, 1] \rightarrow [0, 1]$  increasing  
 $g(t) \leq t$

Increasing transformations can be expressed as neighbourhood models

$$\underline{P}(A) = \begin{cases} g(P_0(A)) & A \neq \mathcal{X} \\ 1 & A = \mathcal{X} \end{cases}$$

### 2. Particular models

**Vertical Barrier:**  $\underline{P}_{\text{VBM}}(A) = \max\{bP_0(A) + a, 0\}$   
 $a \leq 0, b \geq 0, a + b \leq 1$   
 $A \neq \mathcal{X}$

**Linear Vacuous:**  $\underline{P}_{\text{LV}}(A) = (1 - \delta)P_0(A)$   
 $\delta \in [0, 1], A \neq \mathcal{X}$

**Pari Mutuel:**  $\underline{P}_{\text{PMM}}(A) = \max\{(1 + \delta)P_0(A) - \delta, 0\}$   
 $\delta \geq 0$

**Total Variation:**  $\underline{P}_{\text{TV}}(A) = \max\{P_0(A) - \delta, 0\}$   
 $\delta \geq 0, A \neq \mathcal{X}$

Proposal

And... what about  
starting with  $\underline{P}$ ?

### 3. Distortion of lower probabilities

#### Distortion procedure

$\Delta$ : family of distortion  
parameters

$\{\phi_\lambda\}_{\lambda \in \Delta}$ : family of transforming functions  
 $\phi_\lambda: [0, 1] \rightarrow [0, 1]$   
 $\phi_\lambda$  increasing  
 $\phi_\lambda(t) \leq t$

$$\underline{Q}_\lambda[\underline{P}](A) = \begin{cases} \phi_\lambda(\underline{P}(A)) & A \neq \mathcal{X} \\ 1 & A = \mathcal{X} \end{cases}$$

### 4. Particular (imprecise) models

**Imprecise VBM:**  $\underline{Q}_{(a,b)}[\underline{P}](A) = \max\{b\underline{P}(A) + a, 0\}$

**Imprecise LV:**  $\underline{Q}_\delta^{\text{LV}}[\underline{P}](A) = (1 - \delta)\underline{P}(A)$

**Imprecise PMM:**  $\underline{Q}_\delta^{\text{PMM}}[\underline{P}](A) = \max\{(1 + \delta)\underline{P}(A) - \delta, 0\}$

**Imprecise TV:**  $\underline{Q}_\delta^{\text{TV}}[\underline{P}](A) = \max\{\underline{P}(A) - \delta, 0\}$

### 5. Desirable properties

#### Basic properties

##### Expansion (P1):

$$\underline{Q}_{\lambda_1}[\underline{P}] \leq \underline{Q}_{\lambda_2}[\underline{P}] \text{ if } \lambda_1 \geq \lambda_2$$

##### Semigroup (a) (P2a):

$$\underline{Q}_{\lambda_0}[\underline{P}] = \underline{P} \text{ for some } \lambda_0 \leq \lambda \text{ for any } \lambda \in \Lambda$$

##### Semigroup (b) (P2b):

$$\underline{Q}_{\lambda_2 + \lambda_1}[\underline{P}] = \underline{Q}_{\lambda_2}[\underline{Q}_{\lambda_1}[\underline{P}]]$$

##### Structure preservation (P3):

$$\underline{Q}_\lambda[\underline{P}] \text{ preserves the properties (ASL, coherence, ... ) of } \underline{P}$$

##### Reversibility (P4):

$$\underline{P}(A) = \varphi_\lambda(\underline{Q}_\lambda[\underline{P}](A)) \text{ for some } \varphi_\lambda$$

#### Distortion of credal sets

##### Expression as a neighbourhood (P8):

$$\mathcal{M}(\underline{Q}_\lambda[\underline{P}]) = \{Q \mid d(Q, \underline{P}) \leq \mu\}$$

for some function  $d$  comparing probabilities and lower probabilities and  $\mu$  depending on  $\lambda$

##### Extreme points commutativity (P9):

$$\underline{Q}_\lambda[\underline{P}](A) = \inf \left\{ Q(A) \mid Q \in \bigcup_{P \in \text{ext}(\mathcal{M}(\underline{P}))} \mathcal{M}(\underline{Q}_\lambda[\underline{P}]) \right\}$$

##### Strong commutativity (P10):

$$\mathcal{M}(\underline{Q}_\lambda[\underline{P}]) = \bigcup_{P \in \mathcal{M}(\underline{P})} \mathcal{M}(P)$$

Which are the  
desirable properties  
for a distortion  
procedure?

#### Invariance properties

##### Permutations (P5)

$\sigma$ : permutation  
 $A = \{x_{i_1}, \dots, x_{i_k}\}$   
 $A^\sigma = \{x_{\sigma(i_1)}, \dots, x_{\sigma(i_k)}\}$   
 $\underline{P}^\sigma(A) = \underline{P}(A^\sigma)$   
 $\underline{Q}_\lambda[\underline{P}]^\sigma(A) = \underline{Q}_\lambda[\underline{P}](A^\sigma)$

$$\underline{P} \xrightarrow{\sigma} \underline{P}^\sigma$$
$$\underline{Q}_\lambda[\underline{P}] \xrightarrow{\sigma} (\underline{Q}_\lambda[\underline{P}])^\sigma = \underline{Q}_\lambda[\underline{P}^\sigma]$$

##### Marginalisation (P6)

$\Pi$ : partition of  $\mathcal{X}$   
 $\underline{P}^\Pi$ : restriction of  $\underline{P}$   
to  $\mathcal{P}(\Pi)$   
 $\underline{Q}_\lambda[\underline{P}]^\Pi$ : restriction of  
 $\underline{Q}_\lambda[\underline{P}]$  to  $\mathcal{P}(\Pi)$

$$\underline{P} \xrightarrow{\Pi} \underline{P}^\Pi$$
$$\underline{Q}_\lambda[\underline{P}] \xrightarrow{\Pi} (\underline{Q}_\lambda[\underline{P}])^\Pi = \underline{Q}_\lambda[\underline{P}^\Pi]$$

##### Conditioning (P7)

Regular extension  
 $B$  such that  
 $\underline{Q}_\lambda[\underline{P}](B) > 0$   
 $\underline{P}_B = \underline{P}(\cdot|B)$   
 $\underline{Q}_\lambda[\underline{P}]_B = \underline{Q}_\lambda[\underline{P}](\cdot|B)$

$$\underline{P} \xrightarrow{B} \underline{P}_B$$
$$\underline{Q}_\lambda[\underline{P}] \xrightarrow{B} (\underline{Q}_\lambda[\underline{P}])_B = \underline{Q}_\lambda[\underline{P}_B]$$

Results

Which properties  
are satisfied  
by each model?

### 6. Results

Model	(P1)	(P2) (P2a) (P2b)	ASL	Coh.	2-monot.	(P3) $k$ -monot.	(P4)	(P5)	(P6)	(P7)	(P8)	(P9)	(P10)
IVBM	N.A.	N.A.	N.A.	✓✓	✓✓	✓✓	✓ if $b \neq 0$ and $\min_x \underline{P}(\{x\}) \geq -a/b$	✓✓	✓✓	✗	✓✓	✓✓	✓ $\underline{P}$ 2-monot.
ITV	✓✓	✓✓	✓✓	✓✓	✓✓	✓✓	✓ if $\delta \leq \min_x \underline{P}(\{x\})$	✓✓	✓✓	✗	✓✓	✓✓	✓ $\underline{P}$ 2-monot.
ILV	✓✓	✓✓	✗	✓✓	✓✓	✓✓	✓✓	✓✓	✓✓	✗	✓✓	✓✓	✓ $\underline{P}$ 2-monot.
IPMM	✓✓	✓✓	✗	✓✓	✓✓	✓✓	✓ if $\underline{Q}[\underline{P}](A) > 0$ for $A \neq \emptyset$	✓✓	✓✓	✗	✓ if $\underline{P}(A) > 0$ for $A \neq \emptyset$	✓✓	✓ $\underline{P}$ 2-monot.

### 7. Conclusions

- Some properties do not hold for all IVBM...
- ...but hold for specific models
- Remarkably, the ILV *always* preserves  $k$ -monotonicity and reversibility
- Under 2-monotonicity, distorting  $\underline{P}$  is equivalent to distorting  $\mathcal{M}(\underline{P})$
- ❓ is it possible to characterise the distortion through its associated set of almost desirable gambles?
- ❓ are the solutions of a coalitional game preserved by the distortion?

### 8. References

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At a glance