Independence for Imprecise Probability

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Basics

This lecture tries to answer:

What is the right definition of independence for imprecise probabilities?

And the conclusion is:

There are several definitions of independence and there are not a right definition, but several definitions each one applicable in a different situation

Outline

- Unconditional Independence
  - Unkown Interaction
  - Epistemic Irrelevance and Independence
  - Strong Independence
  - Other Concepts (random sets, Kuznetsov).
- Conditional Independence
  - Epistemic Irrelevance and Independence
  - Strong Independence

Probabilistic Independence and Conditional I.

- We say X and Y are independent under P if and only if

$$P(x,y) = P^{\downarrow X}(x).P^{\downarrow Y}(y)$$

- We say X and Y are independent given Z under P if and only if

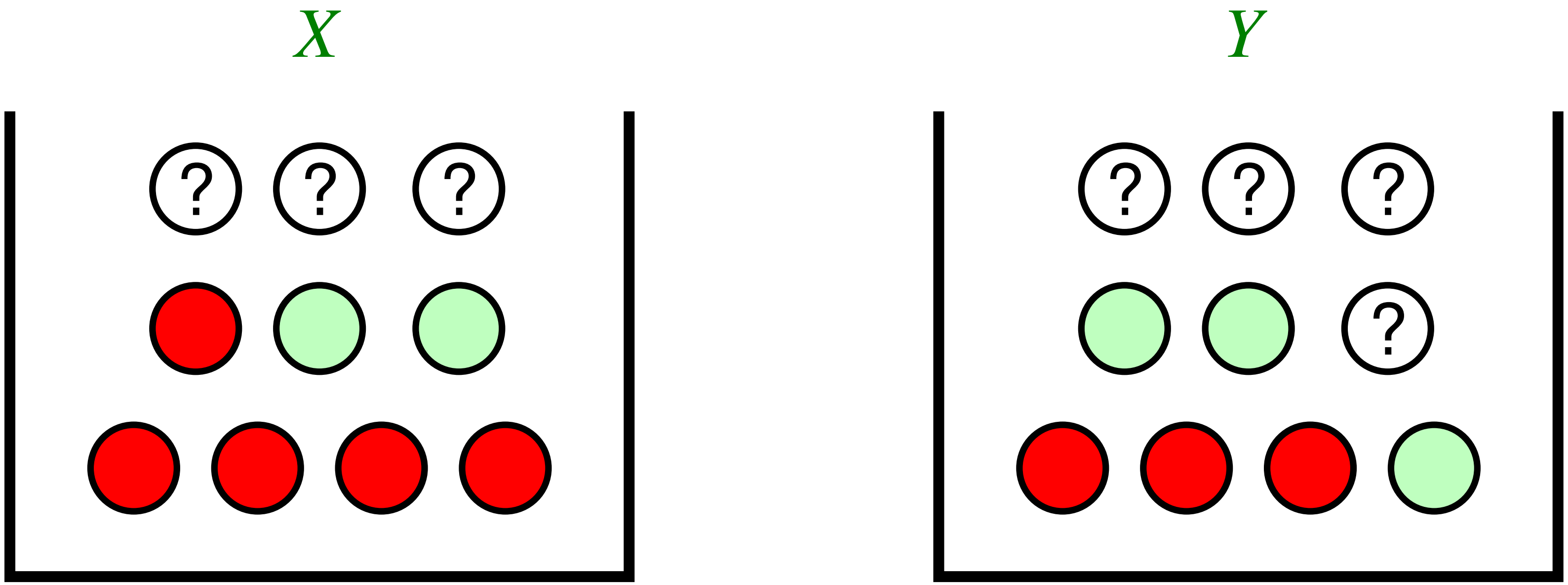
$$P(x,y,z).P^{\downarrow Z}(z) = P^{\downarrow X,Z}(x,z).P^{\downarrow Y,Z}(y,z)$$

Two Basic Approaches

- A basic property (for example all the probabilities in the credal set verify probabilistic definition).
- The joint credal set is the natural extension of the marginal sets under the basic property

We will follow the second option.

Example



Unknown Interaction

- If $\mathcal{M}(X,Y)$ is the joint credal set, then

$$\mathcal{M}(X,Y) = \mathcal{M}(X) \oplus \mathcal{M}(Y) = \{P : P^{\downarrow X} \in \mathcal{M}(X), P^{\downarrow Y} \in \mathcal{M}(Y)\}$$

- If $\mathcal{D}(X,Y)$ is the joint set of acceptable gambles, then

$$\mathcal{D}(X,Y) = \mathcal{D}(X) \oplus \mathcal{D}(Y) = \overline{(\mathcal{D}(X)^{\uparrow X,Y} \cup \mathcal{D}(Y)^{\uparrow X,Y})}$$

the only gambles that are directly judged to be acceptable are gambles which depend on just one of the marginal outcomes.

Properties

- $\mathcal{M}(X,Y)$ is a very large set: It includes probability distributions as
  $P(\{(red,red)\}) = P(\{(green,green)\}) = 0.5$,
  $P(\{(red,green)\}) = P(\{(green,red)\}) = 0.5$.
  Any procedure to draw the ball is allowed as soon the marginals are the same.
- It does not verify the *product rule*

$$\underline{P}(\{(red,red)\}) = 0 < 0.15 = \underline{P}_X(\{red\})\underline{P}_Y(\{red\})$$

$$\underline{P}(A_1 \times A_2) = \max\,\{0, \underline{P}_X(A_1) + \underline{P}_Y(A_2) - 1\}$$

- It produces *dilation*: learning the color of one ball makes the behavior about the other ball more cautious (less acceptable gambles or more possible probability distributions).

Epistemic Irrelevance

We say that the first experiment is *epistemically irrelevant* to the second when

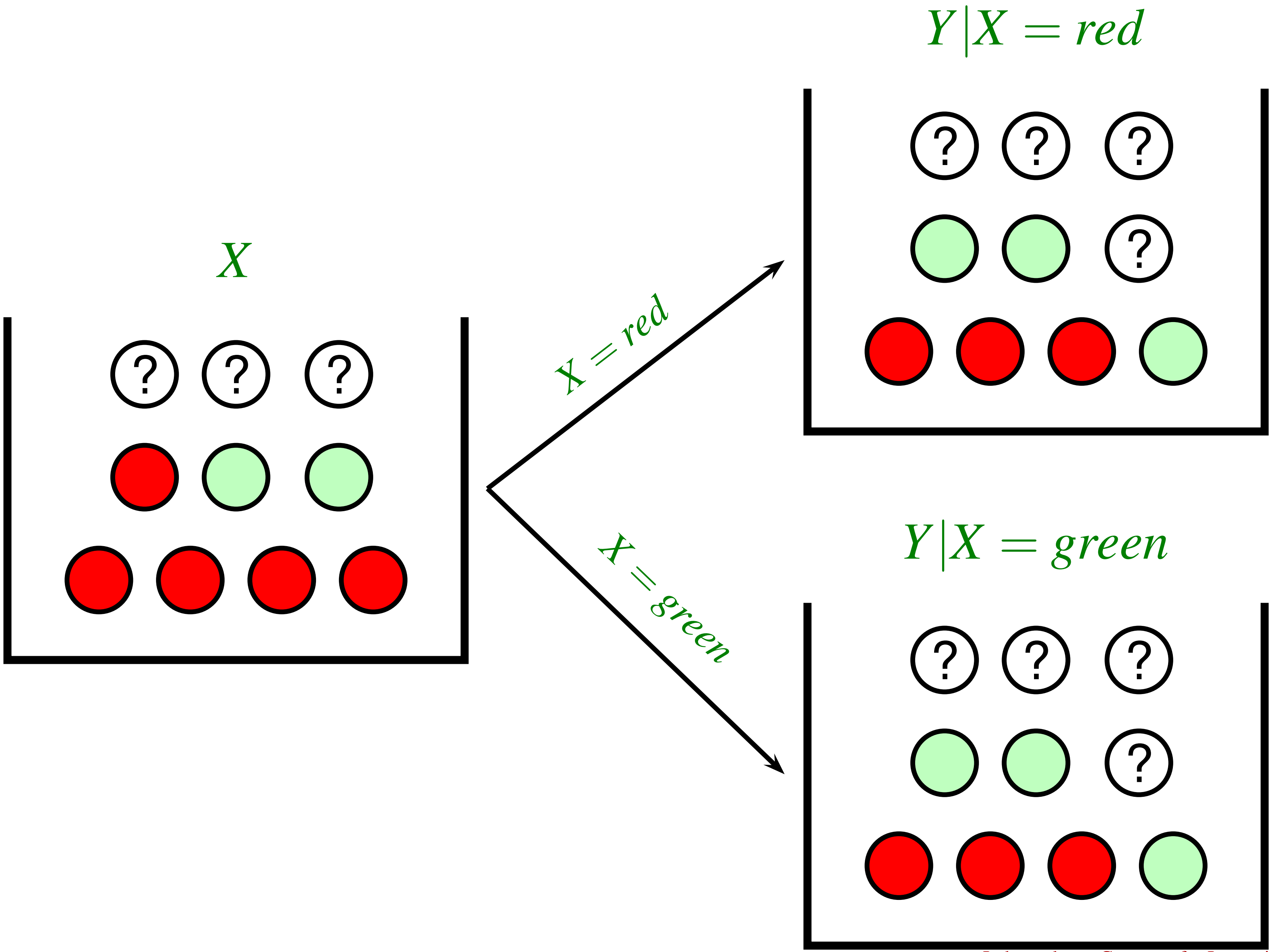
- In terms of gambles: $\mathcal{D}(X,Y)$ is the least informative set of gambles with marginals $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ and such that for every $x \in U_x$ we have

$$\mathcal{D}(Y|X = x) = \mathcal{D}(Y)$$

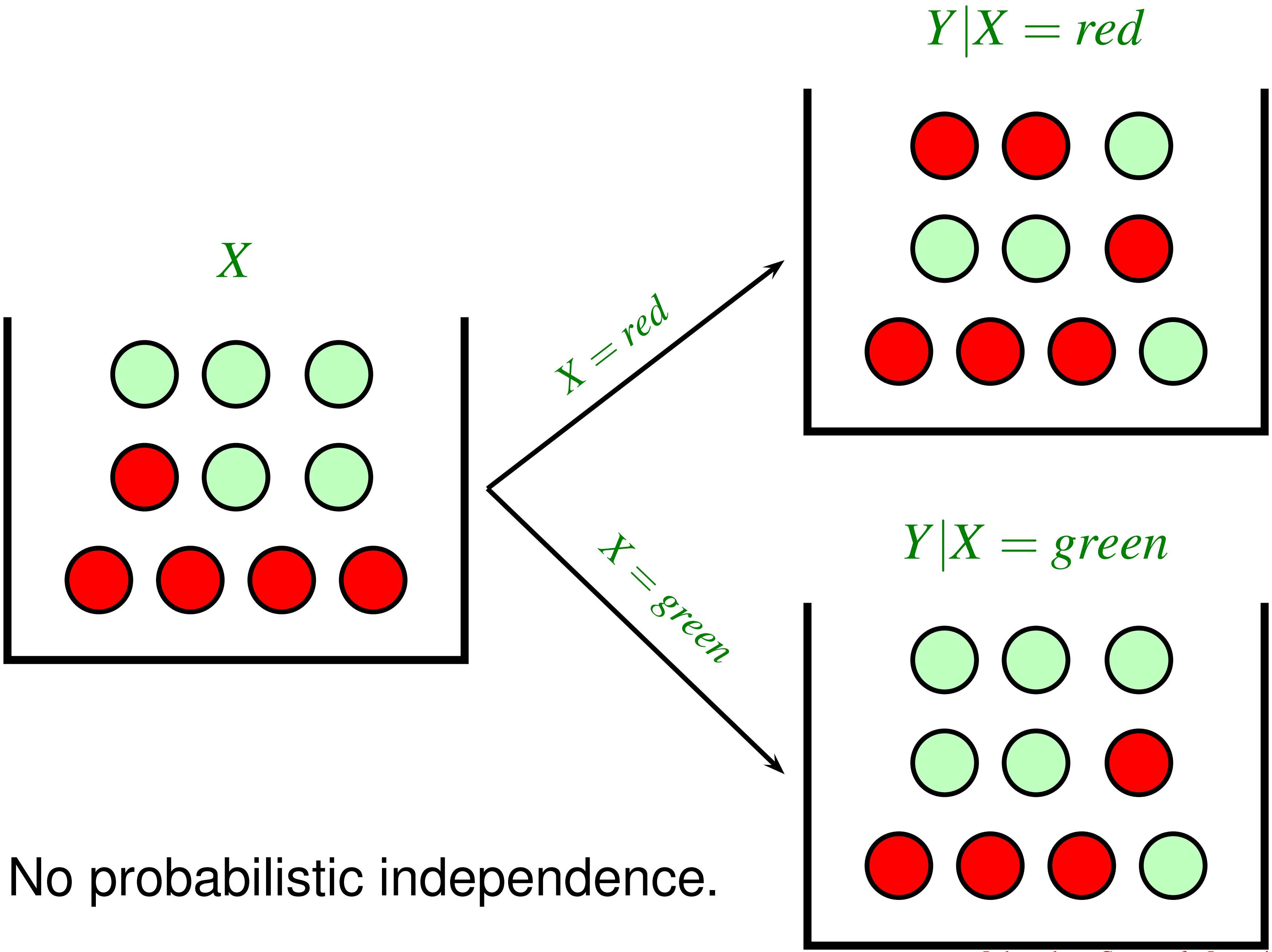
- In terms of probabilities and assuming convexity: $\mathcal{M}(X,Y)$ is the set of probability distributions P given by:

$$P(x,y) = P(x).P_x(y), \qquad P \in \mathcal{M}(X), P_x \in \mathcal{M}(Y) \forall x$$

Example



Example



No probabilistic independence.

Properties

- It is asymmetrical
- It verifies the product rule: $\underline{P}(A_1 \times A_2) = \underline{P}(A_1).\underline{P}(A_2)$
- The conditional lower and upper probabilities for $X = Red$ given the color of the second ball (any of them) are $0.3 - 0.903$ which are wider (*dilation*) than marginal probabilities $0.5 - 0.8$.

Epistemic Irrelevance

- A conditional family $\mathcal{D}_v(Y|X)$ is equal to the family of coherent sets gambles on Y indexed by the values of X.
- Its natural extension to (X,Y) is the set of gambles:

$$\{f : f.I_{X=x} \in \mathcal{D}_v(Y|X=x), \forall x \in U_X\}$$

- The strong extension of $\mathcal{D}(Y)$ to (X,Y) is equal to the natural extension of the family $\mathcal{D}_v(Y|X)$ given by $\mathcal{D}_v(Y|X) = \mathcal{D}(Y)$. It will be denoted by $\mathcal{D}(Y)^{\Uparrow X,Y}$.
- There is epistemic irrelevance if and only if

$$\mathcal{D}(X,Y) = \mathcal{D}(X)^{\uparrow X,Y} \oplus \mathcal{D}(Y)^{\Uparrow X,Y}$$

Epistemic Irrelevance

Epistemic irrelevance of X to Y implies two things:

- The behaviour about Y is given indexed by the values of X; i.e. we give our actitude to Y for each one of the values $x \in U_X$.
- This behaviour is the same for the different values $x \in U_X$.

Epistemic Independence

X and Y are epistemic independent if and only if X is epistemic irrelevant to Y and Y epistemic irrelevant to X.

The independent natural extension is the appropriate model to use when we are given the models $\mathcal{D}_1$ and $\mathcal{D}_2$ for the two marginal experiments, together with a judgement that the experiments are epistemically independent, but we are not willing to make stronger assumptions, e.g., that there are underlying stochastic mechanisms which are stochastically independent

There is epistemic independence if and only if

$$\mathcal{D}(X,Y) = \mathcal{D}(X)^{\Uparrow X,Y} \oplus \mathcal{D}(Y)^{\Uparrow X,Y}$$

Strong Independence

There is strong independence if and only if

$$\mathcal{M}(X,Y) = \{P_X \times P_Y : P_X \in \mathcal{M}_X, P_Y \in \mathcal{M}_Y\}.$$

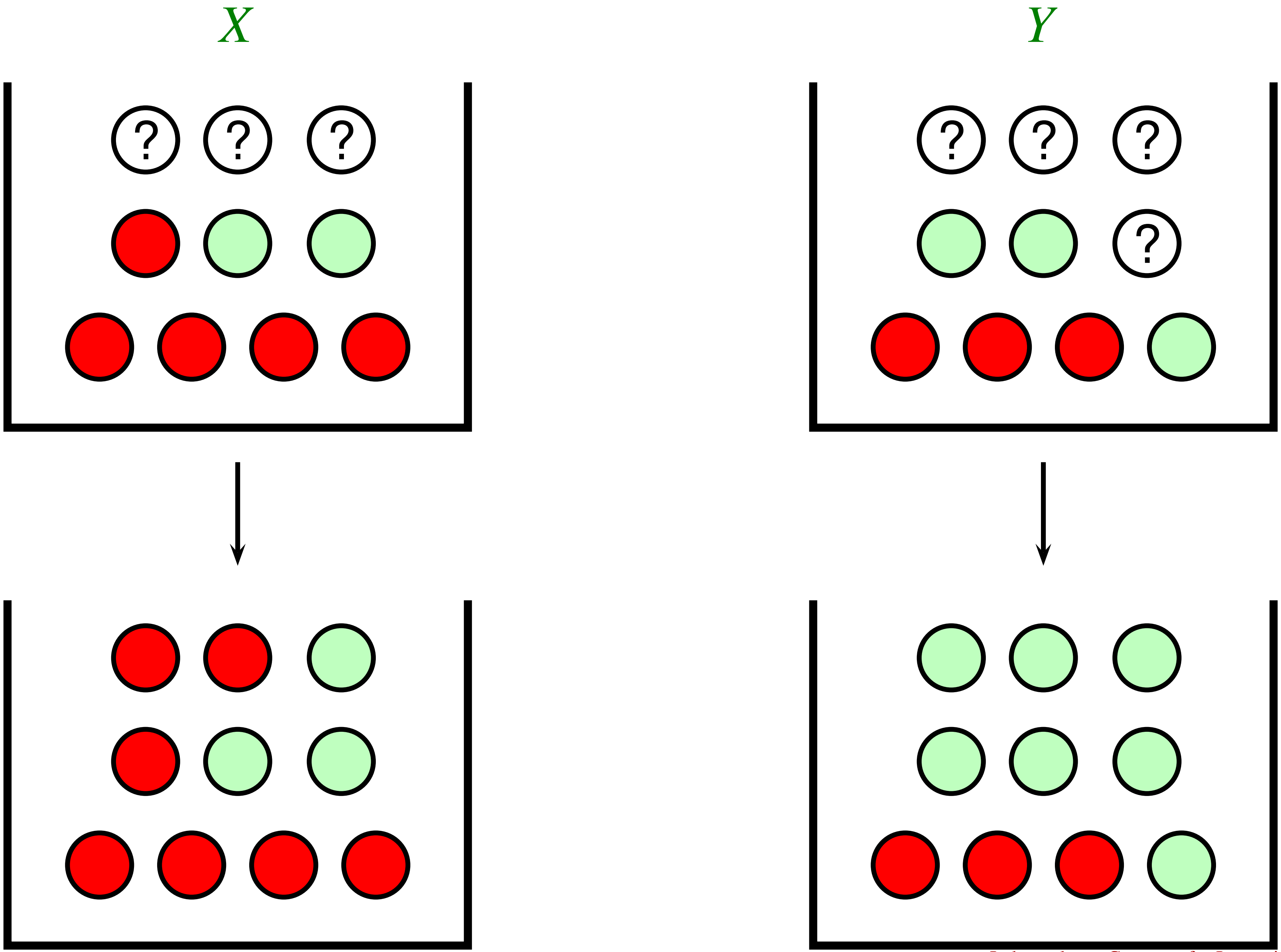
Appropriate under:

(a) the two outcomes result from random experiments, each governed by a unique (but unknown) probability distribution;

(b) the random experiments are stochastically independent,

(c) We do not know of any relationship between the two marginal probability distributions that would enable us to rule out some of the possible combinations of marginal distributions

Example



Properties

- It verifies the factorization $\underline{P}(A_1 \times A_2) = \underline{P}_1(A_1).\underline{P}_2(A_2)$.
- It is appropriate for objective probabilities.
- It is difficult to express in terms of gambles.
- It is the strongest concept. If we have (non-negative) marginal sets $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ and $\mathcal{M}_1(X,Y), \mathcal{M}_2(X,Y), \mathcal{M}_3(X,Y), \mathcal{M}_4(X,Y)$ the joint sets under unknown interaction, epistemic irrelevance, epistemic independence, and strong independence, we have

$$\mathcal{M}_4(X,Y) \subseteq \mathcal{M}_3(X,Y) \subseteq \mathcal{M}_2(X,Y) \subseteq \mathcal{M}_1(X,Y)$$

Other: Random Sets Independence

It is for lower previsions:

$$\underline{P}(A) = \sum_{B \subseteq A} m(B)$$

where m is a mass assignment.

- Marginals: $\underline{P}_1$ and $\underline{P}_2$ which masses m_1 and m_2.

$$m(A_1 \times A_2) = m_1(A_1) m_2(A_2)$$

with $m(A) = 0$ for all subsets of $U_X \times U_Y$ which are not of the form $A = A_1 \times A_2$.

Appropriate if the uncoloured balls are painted after selecting them, by an unknown criterion.

Properties

$$\underline{P}(A_1 \times A_2) = \underline{P}(A_1).\underline{P}(A_2)$$

- It is more precise than unknown interaction, but less than strong independence.
- Even that, we can have dilation.

Example

Intervals for the event S that both balls have the same colour in the urns of examples under the different models of independence.

	$\underline{P}(S)$	$\overline{P}(S)$
Unknown Interation	0.00	1.00
Random Sets Independence	0.21	0.79
Epistemic Irrelevance	0.30	0.70
Epistemic Independence	0.32	0.68
Strong Independence	0.38	0.62

Kuznetsov Independence

The joint lower prevision is the less informative verifying:

$$\underline{P}(f(x).g(y)) = \min \begin{pmatrix} \underline{P}(f).\underline{P}(g) \\ \underline{P}(f).\overline{P}(g) \\ \overline{P}(f).\underline{P}(g) \\ \overline{P}(f).\overline{P}(g) \end{pmatrix}$$

It is less precise than strong extension but more than epistemic independence.
Difficult to handle.

Repetition Independence

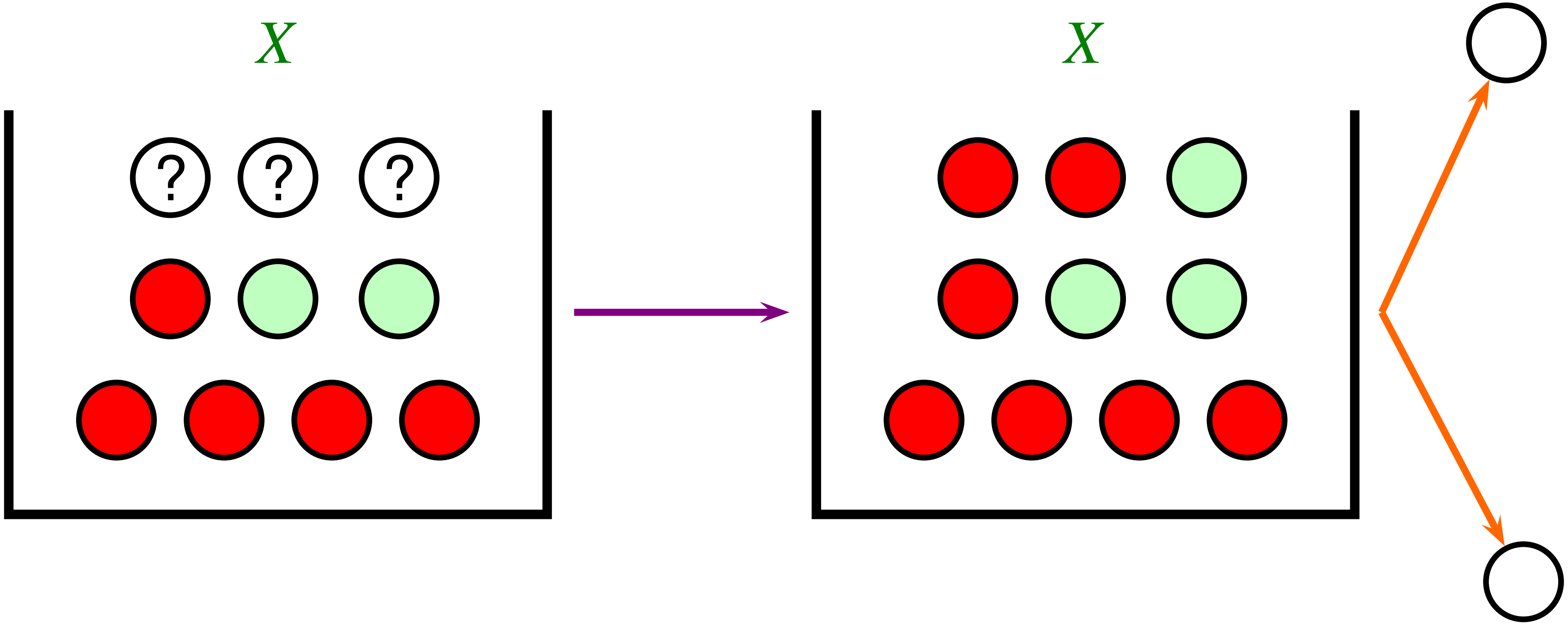
In this case, $U_X = U_Y$, we assume the same marginal set $\mathcal{M}(X) = \mathcal{M}(Y)$.
We have repetition independence when the joint credal set is given by:

$$\mathcal{M}(X,Y) = \{P \times P : P \in \mathcal{M}(X)\}$$

The two marginal probabilities are the same and we have repetitions with independence according to the same distribution (statistical samples).
Convexity in $\mathcal{M}(X)$ is meaningful:

$$\text{Equivalence} + \text{Repetition Independence} \not\Rightarrow \text{Equivalence}$$

Example



Conditional Independence

Intuition $I(X,Y|Z)$ if X and Y are independent when we know the exact value of $Z = z$.

- It is more difficult to handle.
- Even one definition of unconditional independence can give rise to different possible definitions when conditional independence is considered.
- We get asymmetrical concepts even for strong independence.

Epistemic Conditional Irrelevance

Given three variables, X, Y, and Z, we say that X is irrelevant to Y given Z if and only if

$$\mathcal{D}(X,Y,Z) = \mathcal{D}_m(X,Z)^{\uparrow X,Y,Z} \oplus \mathcal{D}_v(Y|Z)^{\Uparrow X,Y,Z}$$

where $\mathcal{D}_m(X,Z)$ is the marginal set on variables (X,Z) and $\mathcal{D}_v(Y|Z)$ is a family of gambles about Y for each value of Z. This definition implies:

- For every $x \in U_X, z \in U_Z$, the set $\mathcal{D}_v(Y|Z=z, X=x)$ is equal to the set $\mathcal{D}_v(Y|Z=z)$
- $\mathcal{D}(X,Y,Z) = \mathcal{D}_m(X,Z) \oplus D_v(Y|Z,X)$

Properties

- Epistemic irrelevance does not verify the symmetry property.
- It is not always the case that $I(X,Y|X)$ as this would imply that $\mathcal{D}(X,Y) = \mathcal{D}_m(X) \otimes \mathcal{D}_v(Y|X) = \mathcal{D}_m(X) \oplus \mathcal{D}_v(Y|X)$. This is true in classical probability but not always in imprecise probability.

Strong Conditional Independence

We will consider the following definition (X is strongly irrelevant to Y given Z)

$$\mathcal{M}(X,Y,Z) = \{p(x,z).q_z(y) \,:\, p \in \mathcal{M}(X,Z), q_z \in \mathcal{M}_v(Y|Z=z)\}$$

where $\mathcal{M}_v(Y|Z)$ is a family of credal sets indexed by the values $x \in U_Z$.

Example

- Assume that we have three urns with 10 balls each.
- The first one, U_1, has 4 red, 4 blue, and 2 of unknown colour, the second, U_2, has 3 red, 5 blue, and 2 unknown, and the third, U_3, 6 red, 2 blue, and 2 of unknown colour.
- We also have that the balls with unknown colour are blue or red and that they have the same composition of colours in the three urns: either are both red, or blue, or one red and the other blue.
- We consider the following experiment: a ball is chosen at random from the first urn, U_1, (its colour is variable Z). Then an urn (U_2 or U_3) is chosen and two balls are drawn at random and with replacement from it (variables X and Y represent the colours of these two balls). If Z is red then both balls are from U_2 and if Z is blue then the balls are from U_3.

X is not strongly irrelevant to Y given Z.

Example

- Assume 5 urns with the following composition:

	U_1	U_2	U_3	U_4	U_5
red	2	1	3	7	8
blue	8	9	7	2	1
unknown	0	0	0	1	1

- The unknown balls in U_4 and U_5 have the same colour in both cases.
- We randomly draw a ball from U_1 (variable Z)
- Selection of X: If Z is red, then we draw a ball from U_2. If Z is blue, then a ball from U_3.
- Selection of Y: If Z is red, then we draw a ball from U_4. If Z is blue, then a ball from U_5

There is not strong irrelevance.

Example

	U_1	U_2	U_3	U_4	U_5
red	2	1	3	7	8
blue	8	9	7	2	1
unknown	0	0	0	1	1

The unknown balls in U_4 and U_5 have the same colour in both cases.

The problem is that the conditional of Y given Z can not be specified separately for the different values of Z.
We have two possibilities:

$$\begin{pmatrix} 0.8 & 0.9 \\ 0.2 & 0.1 \end{pmatrix} \qquad \begin{pmatrix} 0.7 & 0.8 \\ 0.3 & 0.2 \end{pmatrix}$$

But,

$$\begin{pmatrix} 0.8 & 0.8 \\ 0.2 & 0.2 \end{pmatrix}$$

is not possible.

Example

	U_1	U_2	U_3	U_4	U_5
red	2	1	3	7	8
blue	8	9	7	2	1
unknown	0	0	0	1	1

If we remove the condition: *The unknown balls in U_4 and U_5 have the same colour in both cases.*

The conditional of Y given Z can not be specified separately for the different values of Z:

- For $Z = red$ we have that the conditional probability of Y can be: $(0.7, 0.3)$ and $(0.8, 0.2)$.
- For $Z = blue$ we have that the conditional probability of Y can be: $(0.8, 0.2)$ and $(0.9, 0.1)$.

Any combination of these conditional probabilities is possible: There is strong conditional irrelevance.

Conclusions

- There are several concepts. No clear unique solution.
- Epistemic irrelevance: useful for subjective probabilities.
- Strong independence: for physical probabilities.
- Separately specified: important condition for conditional independence.

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