Graphical Models with Imprecise Probability

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Outline

- Basics of Imprecise Probability and Notation (10 min.)
- Independence Concepts (30 min.)
- Testing Independence, Building Classification Trees (30 min.)
- Bayesian Networks. Learning (30 min.)
- Bayesian Networks. Inference (20 min.)

Gambles

- Variables $X,Y,Z,W,\ldots$ taking values on finite sets $U_X,U_Y,U_Z,U_W,\ldots$
- In lowercase x we will represent a generic value of variable X: $x \in U_X$.
- Sets of Variables will be represented in bold $\mathbf{X}$ taking values on finite sets $U_{\mathbf{X}} = \prod_{Y\in\mathbf{X}} U_Y$.
- A generic value of $\mathbf{X}$ will be represented as $\mathbf{x}$.
- A gamble about X is a real function, f, defined on U_X.
- $\mathcal{L}(X)$ is the set of all possible gambles about X.

Sets of Desirable Gambles

Sets of desirable gambles $\mathcal{D}(X)$ should verify the following inference rules (Walley, 1991):

D1. If $f \geq 0$, then $f \in \mathcal{D}(X)$

D2. If $f \in \mathcal{D}(X)$ and $\lambda \geq 0$, then $\lambda.f \in \mathcal{D}(X)$

D3. If $f, g \in \mathcal{D}(X)$, then $f + g \in \mathcal{D}(X)$

Sets verifying these properties will be called closed.

If they also verify that $-1 \notin \mathcal{D}(X)$ they will be called coherent.

Coherent Sets and Credal Sets

- A Credal Set about X is a set of probability measures, $\mathcal{M}(X)$, about X.
- Two credal sets are equivalent if they have the same convex hull.
- A credal set and a set of desirable gambles are compatible if and only if

  $$\forall f, \inf \{\mathsf{E}_P[f] : P \in \mathcal{M}(X)\} = \sup \{\mu : f - \mu \in \mathcal{D}(X)\} = \underline{P}(f)$$

  where $\mathsf{E}_P[f]$ is the mathematical expectation with respect to P.
- $\underline{P}(f)$ is called the lower prevision of f.

Coherent Sets and Credal Sets

- A set of desirable gambles define an unique credal convex set.

$$\mathcal{M}(X) = \{P : \mathsf{E}_P[f] \geq 0, \forall f \in \mathcal{D}(X)\}$$

- $\mathcal{D}_1(X)$ is said to be less informative than $\mathcal{D}_2(X)$ if and only if $\mathcal{D}_1(X) \subseteq \mathcal{D}_2(X)$.
- We can have *different* coherent sets of gambles associated to the same convex set $\mathcal{M}(X)$. The *least informative* one is:

$$\mathcal{D}(X) = \{f : \underline{P}(f) > 0 \text{ or } f \geq 0\}$$

Other possible sets,

$$\mathcal{D}'(X) = \{f : \underline{P}(f) \geq 0\}$$

Operations in Sets of Gambles

- If $\mathcal{R}(X)$ is a set of gambles, then the set of gambles generated by application of properties D1, D2, and D3 (the intersection of all the sets verifying these properties and containing $\mathcal{R}(X)$) will be called the natural extension of $\mathcal{R}(X)$ and denoted by $\overline{\mathcal{R}(X)}$.
- If $B \subseteq U_X$, then the lower (upper) probability of B, $\underline{P}(B)(\overline{P}(B))$, is the lower (upper) prevision of the indicator function I_B of B.
- The marginalization of a closed set of gambles about (X,Y) to X: $\mathcal{D}(X,Y)^{\downarrow X} = \mathcal{D}(X,Y) \cap \mathcal{L}(X)$, where $f \equiv f'$ if $f'(x,y) = f(x)$.

Operations in Sets of Desirable Gambles

- The weak extension of $\mathcal{D}(X)$ to (X,Y): $\mathcal{D}(X)^{\uparrow X,Y}$, it is the natural extension on (X,Y) of $\mathcal{D}(X)$.
- The Combination:
  $\mathcal{D}(X,Y) \oplus \mathcal{D}(Y,Z) = \overline{(\mathcal{D}(X,Y)^{\uparrow X,Y,Z} \cup \mathcal{D}(Y,Z)^{\uparrow X,Y,Z})}$.
- The set of desirable conditional gambles given B is $\mathcal{D}(X|B) = \{f \in \mathcal{L}(X) : f.I_B \in \mathcal{D}(X)\}$, where I_B is the indicator function of B.

Conditioning (Walley)

$U_X = \{x_1, x_2, x_3, x_4\}$ and a credal set with two extreme probability distributions: $p_1 = (0, 0, 0.25, 0.75), p_2 = (0, 0, 0.5, 0.5)$.
Two sets of desirable gambles: $\mathcal{D}_i(X) = \overline{\mathcal{R}_i(X)}, i = 1, 2$, with

$$\mathcal{R}_1(X) = \{f : f(x_3) + 3f(x_4) > 0, f(x_3) + f(x_4) > 0\}$$

$$\mathcal{R}_2(X) = \mathcal{R}_1(X) \cup \{f : f(x_3) = f(x_4) = 0, f(x_1) + f(x_2) > 0\}$$

$\mathcal{R}_1(X)$ implies $p(x_1) = p(x_2) = 0$.
$\mathcal{D}_1(X|\{x_1, x_2\})$ is the vacuous set of gambles.
$\mathcal{D}_2(X|\{x_1, x_2\})$ is the set of desirable gambles generated by $\{f : f(x_1) + f(x_2) > 0\}$ Associated to a single probability distribution: $p(x_1) = p(x_2) = 0.5, p(x_3) = p(x_4) = 0.0$.

Operations in Credal Sets

- The marginalization of a credal set on (X,Y) to X: $\mathcal{M}(X,Y)^{\downarrow X} = \{P^{\downarrow X} : P \in \mathcal{M}(X,Y)\}$, where $P^{\downarrow X}$ is the marginal distribution to X of P.
- The weak extension of $\mathcal{M}(X)$ to (X,Y): $\mathcal{M}(X)^{\uparrow X,Y}$, it is $\{P : P^{\downarrow X} \in \mathcal{M}(X)\}$.
- The Combination: $\mathcal{M}(X,Y) \oplus \mathcal{M}(Y,Z) = \mathcal{M}(X,Y)^{\uparrow X,Y,Z} \cap \mathcal{M}(Y,Z)^{\uparrow X,Y,Z}$.

Conditioning

Given credal set $\mathcal{M}(X)$ we can consider two different definitions of conditioning:

- *Natural extension conditioning*.- It is vacuous if $\underline{P}(B) = 0$ and otherwise defined as

$$\{P(.|B) : P \in \mathcal{M}(X)\}$$

  More appropriate for epistemic probabilities.

- *Regular extension conditioning*.- It is vacuous if $\overline{P}(B) = 0$ and otherwise defined as

$$\{P(.|B) : P \in \mathcal{M}(X), \quad P(B) \neq 0\}$$