Some Exercises on Coherent Lower Previsions

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4th August 2004

1 Probability Measures and Linear Previsions

Suppose we have a probability measure μ defined on the power set $\wp(\mathcal{X})$ of a finite set $\mathcal{X}$. Mathematically, this means that μ is a $\wp(\mathcal{X}) - \mathbb{R}$ map satisfying $\mu(\emptyset) = 0$, $\mu(\mathcal{X}) = 1$, $\mu(A) \geq 0$ for all $A \subseteq \mathcal{X}$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ for any $A, B \subseteq \mathcal{X}$ such that $A \cap B = \emptyset$.

We may interpret these values $\mu(A)$ as fair prices for indicator gambles I_A. This corresponds to the lower prevision $\underline{P}$ defined on $\{I_A \colon A \subseteq \mathcal{X}\} \cup \{-I_A \colon A \subseteq \mathcal{X}\}$ by

$$\underline{P}(I_A) = \overline{P}(I_A) := \mu(A) \tag{1.1}$$

for all $x \in \mathcal{X}$. Recall that we denote $\underline{P}(I_A)$ by $\underline{P}(A)$ and $-\underline{P}(-I_A) = \overline{P}(I_A)$ by $\overline{P}(A)$.

In this exercise, we shall show that natural extension of the lower prevision $\underline{P}$ representing μ coincides with integration with respect to μ.

(a) Preparatory exercise. Show that the integral with respect to μ of gambles $f \in \mathcal{L}(\mathcal{X})$,

$$P_\mu(f) := \int f \,\mathrm{d}\mu = \sum_{x \in \mathcal{X}} \mu(\{x\}) f(x), \tag{1.2}$$

defines a linear prevision on $\mathcal{L}(\mathcal{X})$.

(b) Show that $\underline{P}$ avoids sure loss.

(c) Show that $\underline{P}$ is coherent.

(d) Show that the natural extension $\underline{E}$ of $\underline{P}$ is given by

$$\underline{E}(f) = \int f \,\mathrm{d}\mu = \sum_{x \in \mathcal{X}} \mu(\{x\}) f(x). \tag{1.3}$$

(e) Extra exercise. Show that natural extension defines a one-to-one correspondence between probability measures μ on $\wp(\mathcal{X})$ and linear previsions Q on $\mathcal{L}(\mathcal{X})$. Hence, through coherence, linear previsions are uniquely determined by their values on singletons whenever $\mathcal{X}$ is a finite set.

(f) Another extra exercise. Show that even when $\mathcal{X}$ is not a finite set, there still is a one-to-one correspondence between finitely additive probability measures μ on $\wp(\mathcal{X})$ and linear previsions Q on $\mathcal{L}(\mathcal{X})$. Hint: first stick to simple gambles, then use continuity of linear previsions with respect to the topology of uniform convergence. Hence, a linear prevision on $\mathcal{L}(\mathcal{X})$ is uniquely determined by its value on events (subsets of $\mathcal{X}$).

(g) Yet another extra exercise (non-additive measures). Suppose we have a 2-monotone measure μ defined on the power set $\wp(\mathcal{X})$ of a finite set $\mathcal{X}$, that is, $\mu(\emptyset) = 0$, $\mu(\mathcal{X}) = 1$, $\mu(A) \geq 0$ for all $A \subseteq \mathcal{X}$, and

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B) \tag{1.4}$$

for any $A, B \subseteq \mathcal{X}$. We may interpret these values $\mu(A)$ as supremum buying prices for indicator gambles I_A. This corresponds to the lower prevision $\underline{P}$ defined on $\{I_A \colon A \subseteq \mathcal{X}\}$ by $\underline{P}(I_A) := \mu(A)$ for all $x \in \mathcal{X}$. Show that the lower prevision $\underline{P}$ representing μ is coherent, and that the natural extension of $\underline{P}$ coincides with $\mathrm{C}\int \cdot \,\mathrm{d}\mu$, the Choquet integral with respect to μ. Hint: first show that the Choquet integral defines a coherent lower prevision (use the sub-additivity theorem, which says that $\mathrm{C}\int (f + g)\,\mathrm{d}\mu \geq \mathrm{C}\int f \,\mathrm{d}\mu + \mathrm{C}\int g \,\mathrm{d}\mu$).

Note that the Choquet integral of a gamble f on a finite set $\mathcal{X}$ can be constructed as follows. Since $\mathcal{X}$ is finite, without loss of generality we can write f as

$$f = \alpha_0 + \sum_{i=1}^{n} \alpha_i I_{A_i} \tag{1.5}$$

with $\alpha_0 \in \mathbb{R}$, $\alpha_1 > 0$, $\alpha_2 > 0, \ldots, \alpha_n > 0$ and $A_1 \supset A_2 \supset \cdots \supset A_n$ (where $A \supset B$ means $A \supseteq B$ and $A \neq B$). In terms of α_i's and A_i's, the Choquet integral of f is simply given by

$$\mathrm{C}\int f \,\mathrm{d}\mu := \alpha_0 + \sum_{i=1}^{n} \alpha_i \mu(A_i). \tag{1.6}$$

2 Vacuous Lower Previsions

Let A be a non-empty subset of a (not necessarily finite) set $\mathcal{X}$. Say we only know that the lower probability of A is equal to 1. This assessment is embodied through the lower prevision $\underline{P}$ defined on the singleton $\{I_A\}$ by $\underline{P}(A) = 1$ (again, recall that we denote $\underline{P}(I_A)$ by $\underline{P}(A)$).

(a) Preparatory exercise. Show that the vacuous lower prevision relative to A, defined by

$$\underline{P}_A(f) := \inf_{x \in A} f(x) \tag{2.1}$$

for any $f \in \mathcal{L}(\mathcal{X})$, is a coherent lower prevision on $\mathcal{L}(\mathcal{X})$.

(b) Show that $\underline{P}$ avoids sure loss.

(c) Show that $\underline{P}$ is coherent.

(d) Prove that the natural extension $\underline{E}$ of $\underline{P}$ is equal to the vacuous lower prevision relative to A:

$$\underline{E}(f) = \underline{P}_A(f) = \inf_{x \in A} f(x), \tag{2.2}$$

for any $f \in \mathcal{L}(\mathcal{X})$.

(e) Extra exercise. Each one of the questions (b), (c) and (d) can be solved in three different ways, either using

(i) the primal form—combinations of desirable gambles,

(ii) the dual form—sets of probability measures, or

(iii) the properties of coherence and natural extension, invoking the result proven in the preparatory exercise (a).

Invoke each one of the methods (i), (ii) and (iii) to answer each one of the questions (b), (c) and (d). You may cheat when solving question (d) using method (ii): it is much easier if you assume that $\mathcal{X}$ is finite.

3 P-Boxes

Let $\mathcal{X} = \mathbb{R}$. Let $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. Consider the linear previsions $\underline{P}_{x_1}$ and $\underline{P}_{x_2}$ defined by

$$\underline{P}_{x_1}(f) := f(x_1), \tag{3.1}$$

$$\underline{P}_{x_2}(f) := f(x_2), \tag{3.2}$$

for all $f \in \mathcal{L}(\mathcal{X})$. Note that these linear previsions are vacuous lower previsions relative to singletons. The lower envelope $\underline{P}$ of $\underline{P}_{x_1}$ and $\underline{P}_{x_2}$ is nothing but the vacuous lower prevision relative to the pair $\{x_1, x_2\}$:

$$\underline{P}(f) = \min\{f(x_1), f(x_2)\}. \tag{3.3}$$

Note that $\underline{P}$ is coherent.

(a) Draw the p-box that corresponds to $\underline{P}$.

(b) Prove that the "natural extension" of this p-box, that is, the lower envelope $\underline{E}$ of all linear previsions $Q \in \mathcal{P}(\mathcal{X})$ whose cumulative distribution function

$$F_Q(x) = Q(\{y \in \mathcal{X} \colon y \leq x\}) \tag{3.4}$$

belongs to this p-box, is dominated by the vacuous lower prevision relative to the *interval* $[x_1, x_2]$, that is,

$$\underline{E}(f) \leq \underline{P}_{[x_1,x_2]}(f) \text{ for any gamble } f \in \mathcal{L}(\mathcal{X}). \tag{3.5}$$

What does this mean?

(c) Extra exercise. If you are fond of ϵ's, show that

$$\underline{E}(f) = \sup_{\epsilon>0} \underline{P}_{[x_1-\epsilon,x_2]}(f) \text{ for any gamble } f \in \mathcal{L}(\mathcal{X}). \tag{3.6}$$

4 The Fréchet Bounds

Assume A and $B \subseteq \mathcal{X}$ are logically independent events: $A \cap B$, $A^c \cap B$, $A \cap B^c$ and $A^c \cap B^c$ are non-empty ($\cdot^c$ denotes complementation in $\mathcal{X}$). We assess lower and upper probabilities for A and B, embodied through a lower prevision $\underline{P}$ defined on the set of gambles

$$\mathcal{K} = \{I_A, -I_A, I_B, -I_B\}. \tag{4.1}$$

Recall that we denote $\underline{P}(I_A)$ by $\underline{P}(A)$ and $-\underline{P}(-I_A) = \overline{P}(I_A)$ by $\overline{P}(A)$, and similar for B. Also recall that $\underline{P}(A^c) = 1 - \overline{P}(A)$, because $\underline{P}(1 - I_A) = 1 - \overline{P}(I_A)$.

(a) Preparatory exercise. Consider the case $\underline{P}(A) = \overline{P}(A) = p \in [0, 1]$ and $\underline{P}(B) = \overline{P}(B) = q \in [0, 1]$. Find a one-dimensional parametrisation of the set of probability measures, defined on the algebra generated by A and B, which are compatible with $\underline{P}$. Recall that a probability measure μ is compatible with a lower prevision if for all gambles f in the domain of $\underline{P}$ it holds that f is integrable with respect to μ, and

$$\int f \, \mathrm{d}\mu \geq \underline{P}(f). \tag{4.2}$$

(b) Preparatory exercise (continued). Again consider the case $\underline{P}(A) = \overline{P}(A) = p \in [0, 1]$ and $\underline{P}(B) = \overline{P}(B) = q \in [0, 1]$. Using the one-dimensional parametrisation of the above exercise, characterise the set $\mathcal{M}(\underline{P})$ of linear previsions on $\mathcal{L}(\mathcal{X})$ that dominate $\underline{P}$. Show that $\mathcal{M}(\underline{P})$ is non-empty, and hence, that $\underline{P}$ avoids sure loss. Finally, show that $\underline{P}$ is coherent.

(c) Now we move on to the general case. Show that $\underline{P}$ avoids sure loss if and only if $\underline{P}(A) \leq 1$, $\underline{P}(B) \leq 1$, $\overline{P}(A) \geq 0$, $\overline{P}(B) \geq 0$, $\underline{P}(A) \leq \overline{P}(A)$ and $\underline{P}(B) \leq \overline{P}(B)$.

(d) Show that $\underline{P}$ is coherent if and only if

$$\begin{aligned} &0 \leq \underline{P}(A) \leq \overline{P}(A) \leq 1 \quad \text{and} \\ &0 \leq \underline{P}(B) \leq \overline{P}(B) \leq 1. \end{aligned} \tag{4.3}$$

(e) Assume $\underline{P}$ satisfies Eq. (4.3). Derive the Fréchet bounds by showing that the natural extension $\underline{E}$ of $\underline{P}$ satisfies

$$\begin{aligned} \underline{E}(A \cap B) &= \max\{\underline{P}(A) + \underline{P}(B) - 1, 0\}, & \underline{E}(A \cup B) &= \max\{\underline{P}(A), \underline{P}(B)\}, \\ \overline{E}(A \cap B) &= \min\{\overline{P}(A), \overline{P}(B)\}, & \overline{E}(A \cup B) &= \min\{\overline{P}(A) + \overline{P}(B), 1\}. \end{aligned} \tag{4.4}$$

5 The Three Prisoners Problem

Three men, a, b and c, are in jail. Prisoner a knows that only two of the three prisoners will be executed, but he doesn't know who will be spared. He only knows that all three prisoners have equal probability $\frac{1}{3}$ of being spared. To the warden who knows which prisoner will be spared, a says, "Since two out of the three will be executed, it is certain that either b or c will be. You will give me no information about my own chances if you give me the name of one man, b or c, who is going to be executed." Accepting this argument after some thinking, the warden says, "Prisoner b will be executed."

Does the warden's statement truly provide no information about the chance of a to be executed? We try to solve this problem using the theory of lower previsions.

(a) Let the variable X denote the prisoner that will be spared. Since all three prisoners have equal probability $\frac{1}{3}$ of being spared, we have a prior prevision specified by $\underline{P}_0(\{a\}) = \underline{P}_0(\{b\}) = \underline{P}_0(\{c\}) = \overline{P}_0(\{a\}) = \overline{P}_0(\{b\}) = \overline{P}_0(\{c\}) = \frac{1}{3}$. In a previous exercise, we have shown that the natural extension of $\underline{P}_0$ is given by

$$\underline{E}_0(f) = \frac{1}{3}\Big(f(a) + f(b) + f(c)\Big). \tag{5.1}$$

for any $f \in \mathcal{L}(\mathcal{X})$.

(b) Let the variable Y denote the prisoner named by the warden. Since the warden will not name a, we know that if $X = a$, then Y will be b or c, if $X = b$ then $Y = c$ and if $X = c$ then $Y = b$. Such information is modelled by vacuous conditional lower previsions, again, as described in one of the previous exercises:

$$\underline{P}(g|X = a) = \min\{g(b), g(c)\} \tag{5.2}$$
$$\underline{P}(g|X = b) = g(c) \tag{5.3}$$
$$\underline{P}(g|X = c) = g(b) \tag{5.4}$$

for any gamble $g \in \mathcal{L}(\mathcal{Y})$. Note that in case $X = a$, we do not know the mechanism by which the warden names either b or c for Y. Therefore, it seems appropriate to model this situation through a vacuous lower prevision relative to $\{b, c\}$.

(c) Combine the lower previsions $\underline{E}_0(\cdot)$ on $\mathcal{L}(\mathcal{X})$ and $\underline{P}(\cdot|X)$ on $\mathcal{L}(\mathcal{Y})$, using the marginal extension theorem, to a coherent lower prevision $\underline{E}$ on $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$.

(d) Apply the generalised Bayes rule to calculate $\underline{E}(X = a|Y = b)$, $\overline{E}(X = a|Y = b)$ and $\underline{E}(X \neq a|Y = b)$, $\overline{E}(X \neq a|Y = b)$.

(e) Extra exercise. After naming prisoner b as one of the prisoners to be executed, the warden thinks a little more and decides to play the following slightly sadistic game with prisoner a. The warden continues: "Are you really sure that I have given you no information at all by naming b? If you want to, for a reasonable fee I can arrange your fate to be switched with the fate of prisoner c. Of course, since I have not given you any information at all, you might not care about such arrangement. On the other hand, switching with prisoner c might just save your life... It's up to you to decide!"

Assume the utility of your life is equal to 25,000,000 Cuban Peso and the bribe requested by the warden is 25,000 Cuban Peso. Assuming that the warden really tells the truth about being able to arrange the switch, what would you do if you were prisoner a? (If the value of the bribe is zero, this game is isomorphic to the Monty Hall puzzle, as for instance described in de Cooman & Zaffalon, "Updating beliefs with incomplete observations", Artificial Intelligence, 2004 (in press).)